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ABSTRACT

This document reports the present status of the Comprehensive School Mathematics Program (CSMP). Three major components are described: (1) the criteria used for content selection and the role of staff and consultant mathematicians in its analysis; (2) the rationale for choosing the activity packages, organizational plans for producing and testing the materials, and preliminary results of the pilot testing of Grade 3 packages; and (3) the objectives, structure and production of a course intended for the upper 20 percent of the high school population, which stresses the use of formal logical language. An appendix contains a sample mathematician's outline, a sample activity sequence outline, and a sample activity from the Activity Package Program. A second extensive appendix details the contents of the books for the Elements of Mathematics Program. Other appendices list staff, conferences and materials produced to date. (MM)

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Comprehensive School Mathematics Program

Basic Program Plan

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COMPREHENSIVE SCHOOL MATHEMATICS PROGRAM

BASIC PROGRAM PLAN

SPRING 1970

CENTRAL MIDWESTERN REGIONAL EDUCATIONAL LABORATORY, INC.
St. Louis

COMPREHENSIVE SCHOOL MATHEMATICS PROGRAM

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PREFACE

The Comprehensive School Mathematics Program (CSMP) was envisioned in 1963 as a response to a research impetus in mathematics education manifested in the reports of prominent committees of mathematicians and mathematics educators from the United States and abroad.[1]

CSMP is the result of several years of planning and of some preliminary experimental writing and teaching. The main planning activity originated from a series of conferences of leaders in mathematics education.[2]

In 1966 CSMP was established and became affiliated with Southern Illinois University. A proposal for a long-range curriculum development project was written during this year. The proposal was presented to the Central Midwestern

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[1] Goals for School Mathematics - A Report of the Cambridge Conference on School Mathematics, Houghton Mifflin Co., Boston, 1963. Synopses for Modern Secondary School Mathematics - Organization for Economic Co-operation and Development (OECD), 1961. Mathematics Today, (OECD), 1963.

[2] Foster, Garrett; Kaufman, Burt; Fitzgerald, William, A First Step Towards the Implementation of the Cambridge Mathematics Curriculum in a K-12 Ungraded School, Nova High School, Ft. Lauderdale, Florida, 1966.

Regional Educational Laboratory (CEMREL), and CSMP was incorporated as one of its major programs in the spring of 1967.[3]

From that time, through June 1970, CSMP has been involved in the feasibility phase of its program, in a restructuring of its development cycle, and in a realistic reappraisal of its long-range plans. This paper is a basic program plan for the next five years, with specific emphasis on 1970-71. It takes into account the total history of the program, with its successes and its failures.

The paper is organized into three major sections; the first describes the factors that influenced CEMREL's selection of CSMP and the program's expected outcomes; the second section details the strategies to be employed in the various components of CSMP; the final section gives a summary of activities for the current year and for the next 5 years of the program, with special emphasis, again, on the year 1970-71.

There is no need here to describe or document the "current explosive penetration of mathematical methods into other disciplines, amounting to a virtual 'mathematization of culture'." This has been done eloquently in the recent definitive report on the mathematical sciences issued by the National Academy of Sciences.[4] It is now widely recognized, to further quote from the report, that "the mathematization of our society brings with it an increasing need for people to be able to understand and use mathematics." It is also recognized that "a rising level of mathematical literacy [is] a national objective."

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[3] Southern Illinois University Comprehensive School Mathematics Project. Proposal for Phase I of a Project to Create an Individualized Mathematics Curriculum in the Spirit of the Cambridge Conference Recommendations. Southern Illinois University, Carbondale, Illinois, 1966.

Assuming the urgent need for improving mathematics education, this plan identifies certain deficiencies and describes a program for correcting them.

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[4] The Mathematical Sciences: A Report, National Academy of Sciences, Washington, D.C., 1968.

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I

INTRODUCTION

This section describes
the *factors that influenced*
CEMREL's selection of
CSMP, and the program's
expected outcomes.

- Dissatisfaction with Current Mathematics Education
- History of Experimental Mathematics Programs
- Expected Outcomes of CSMP
- Strategies for Reaching the Expected Outcomes

INTRODUCTION

Each new project designed to change curriculum and instruction in school mathematics has as its motivation the desire to improve certain aspects of current educational practice with which there is deep dissatisfaction.

DISSATISFACTION WITH CURRENT MATHEMATICS EDUCATION

During the past 15 years dissatisfaction with the typical learning situation in American mathematics classrooms has been growing on at least four counts:

1. Irrelevant content of curriculum. Some of the mathematics still being taught in the schools has little justification other than that it is part of the traditional program. Furthermore, it is still organized in the tradition of separation of topics: arithmetic, algebra, geometry, trigonometry, analytic geometry, calculus. As a result, some of the content is irrelevant to the modern uses of mathematics; also, there is little space included in the curriculum for topics such as probability, combinatorics, statistics, topology

and others. (See Component I, described on pp. 23-37.) Although a great deal of the traditional content is still important to modern mathematics, the practice of separating it into distinct compartments obscures the basic unity of mathematics. Not only does separation of topics give a distorted view of mathematics as isolated sets of facts, but it is also inefficient in the sense that the deepest insight into an idea in one "compartment" often comes by viewing it from the standpoint of a different "compartment". For example, a child could get a clearer notion of multiplication of rational numbers by considering it in the context of areas of rectangles. Yet, despite the modern trend to unify mathematics, one still finds a whole year of the typical American secondary program devoted strictly to synthetic plane geometry, with a stubborn refusal to use unifying ideas from algebra and theory of functions. To ignore the unity of mathematics, the interplay between its concepts, is to miss the power of mathematics.

2. Static view of mathematics. Besides the dissatisfaction with the mathematical content of the

school curriculum, there is an increasing dissatisfaction with the methods used to teach it. Many pupils are still being taught to react in rote fashion to a static set of mathematical symbols that to them are devoid of meaning. Pupils too often are expected to learn passively lists of facts and manipulations rather than participate actively in search for meaning and structure within mathematics. It is clear that mathematics cannot be entirely "discovered" by students, but their training should help them relate mathematics to real situations and to develop some facility in mathematizing these situations.

A young adult who has learned some mathematics by rote and who views mathematics merely as a set of formulas that apply to fixed situations will not find his training of much use. Worse, he will probably look back on his mathematics classes as dull and, in too many cases, hateful.

3. Regimented classrooms and materials. Not only are the curriculum and the teaching methods under fire, but the physical setting is considered unsatisfactory. It is not unusual to find 40 pupils in fixed seats in a room, all expected to concentrate on the same

problem or topic at the same time, all to be thinking at the same level of abstraction, and with the same degree of success expected from each one. Thousands of students study the same commercial textbook under the implicit assumption that the development of ideas, the levels of difficulties, and the types of explanations (ranging from concrete to abstract) used in the books are adequate for all these students. In many classes the textbook is the only learning medium available, other than the oral explanations offered by the teacher. Given the wide range of student learning characteristics, even in a "homogeneous" class, such a setting must result either in failure to keep the pace at one end of the scale, or loss of interest at the other, or both. Such a setting forces students with widely varying interests and abilities to conform to a single way of learning and thinking about mathematics, instead of adapting the setting to the learning styles of the students. As a consequence, we are far short of an optimum setting for the realization of the potentials of individual students. The resulting waste of human resources is a major reason for dissatisfaction with current educational systems.

4. Inadequately trained teachers. Since the typical classroom setting is organized around a teacher, his role is crucial in any program designed to improve the deficiencies noted above. In general, teachers have managed to keep the school systems operative in the face of overwhelming restrictions imposed by the established school setting. Unfortunately, even if such restrictions were removed, there probably would not be much improvement in mathematics education. The sad fact is that too few teachers have received the training they need, and want, to cope with innovative curriculum changes. It is often observed that children have less trouble with sophisticated mathematical ideas than do their teachers.

HISTORY OF EXPERIMENTAL MATHEMATICS PROGRAMS

In the 1950's there were several pioneer projects that developed as reactions to the rising awareness of dissatisfactions 1 and 2 listed above. The Maryland Project, SMSG, UICSM, and other projects enlisted professional mathematicians and mathematics educators to prepare new and more relevant teaching materials and to propose new instructional methods. These projects wisely accepted existing school settings, such as teacher-centered classes, single-textbook medium of instruction, and separation of subjects by grades. These projects concentrated on the problem of writing sample textbooks that would illustrate how a transition could begin (for college-bound students) from an unsatisfactory curriculum to one that would eventually be more relevant to the mathematics of today. Even this modest break with the traditional curriculum, described as the "new math", caused a veritable shock wave.

In a few years the desired effect was noticed to some extent in commercial textbooks being written, and the gradual improvement to date in the mathematical content of school courses is undoubtedly due to the fact that the early projects set reasonable goals for improvement.

By the early and middle 60's the educational climate was ready for the next step in attacking dissatisfaction areas 1 and 2 (irrelevant content, static view of mathematics) and for making a beginning in areas 3 and 4 (regimented classrooms and materials, inadequately trained teachers). A distinguished group of mathematicians and scientists, at a conference in 1963, proposed new goals for school mathematics.[1] Their report pleaded for continued efforts in the improvement of mathematics education. SMSG began a second round of sample texts for the secondary school; UICSM shifted its attention from gifted students to slow learners; new projects concentrated on an integrated curriculum for gifted students, and other projects on materials and methods for disadvantaged pupils. At the same time some experimentation began that was designed to improve the education setting through ungraded classes, team teaching, programmed instruction, self-pacing, etc. Within each of these educational models ways were sought of adapting materials and modes of instruction to the needs of individual children, with a concomitant change in the role of teachers.

It is significant to note here that current projects are addressed primarily either to dissatisfaction areas 1 and 2, but not 3 and 4, or to 3 and 4, but not 1 and 2. This is not surprising since the educational climate, until recently, was cool even to a thrust at any one of these areas, much less at all of them simultaneously.

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[1] Goals for School Mathematics - A Report of the Cambridge Conference on School Mathematics, Houghton Mifflin Co. Boston, 1963.

Thanks to the influence of such specifically directed projects, it is felt that the climate is now ready for a coordinated attack on all four areas of dissatisfaction. The CSMP staff believes that, by tackling the total set of problems in mathematics education, a quantum jump can be achieved.

EXPECTED OUTCOMES OF CSMP

When operations began in 1967 under the sponsorship of CEMREL, the CSMP staff felt that its goals needed to be kept as reasonable with respect to the present educational setting as the pioneering projects had done with respect to the setting of the 50's. Yet the educational climate is such that many of the boundary conditions restraining the early experimental projects are conditions that many modern schools are now willing to remove. It is also assumed that funding agencies are ready to remove some of their restrictions, such as the need to produce and disseminate a tangible product in a short period of time.

Stated quite broadly, the ultimate goal of CSMP is to develop mathematics curricula for students from ages 5 through 18 which provide for each student a program sound in content, enjoyable, more appropriate for his needs and abilities, and presented so as to maximize his success in the realization of his potential for learning and using mathematics.

In order to make these broad goals meaningful there are two questions to be answered: 1) What criteria can be used to decide whether mathematics is sound, enjoyable and appropriate? In fact, what do these words mean? 2) The broad goal of maximizing the success in the realization of each student's potential for learning implies some form of individualization of instruction. What goals for individualization are reasonable? How far can a project hope to move in the direction of adapting curricula to students' learning characteristics?

Before answering the first question, let us list some general aims of instruction in mathematics:

1. We want the student to understand that mathematics is a discipline with a long history, that it is fast growing and changing, and has been for a long time too extensive for any one person

to master in detail. We want him to know some of its history as it concerns major contributors and the development of important mathematical concepts.

2. We want the student to enjoy mathematics, to find it worthy of study and effort, to find motivation within, as well as outside, the subject.
3. We want the student to understand that mathematics is useful, in direct pragmatic application, as a tool in other disciplines, and that it is a rich source of models and methods that may be used in a wide variety of ways. We want him to have first-hand experience with such uses, and be capable of using it himself.
4. We want the student to learn how to obtain access to mathematics, how to find what he needs, how to go about using what he finds.
5. We want the student to see, by example and direct experience, that mathematics is a creative enterprise involving a free-wheeling kind of thinking and the utmost of imagination, but that the good intuitive leap or flash of insight comes to the deserving person, namely the one who has developed a solid background. We want him to discover that his intuition can improve with use.

6. We want the students to develop assurance in the soundness of their work; to develop the resources to check their own work independently of outside authority, whether in problem solving or in writing proofs.
7. We want the student to be equipped with the usual mathematical skills and competence for everyday tasks and, in particular, the skills and techniques he needs for the courses he is pursuing concurrently.

In answering question 1 (concerning criteria for deciding whether content is appropriate) one must also take into account the different views of mathematics held by various people. The creative research mathematicians view it in various ways, depending on their fields of research. There are various users of mathematics (scientists, engineers, social scientists, businessmen, etc.) who view mathematics in other ways; mathematics educators are concerned with curriculum and methods, while school administrators, parents, students, and the man in the street view it in terms of the exigencies of their situations and in terms consistent with their backgrounds. Each has needs that some form of mathematics can fulfill. Each can enhance his services to society by bringing to his vocation an "appropriate" background in mathematics. It should be noted that some opponents of the "modern math" accuse it of being directed toward the viewpoint of the research mathematician to the exclusion of the other views of mathematics.

An assumption accepted by CSMP is that all of these views of mathematics are viable and valid. One goal for the content of CSMP is a curriculum so designed that none of these views is excluded; students at each stage of their schooling should have, therefore, a maximum number of avenues open for their adult uses of mathematics. Such a curriculum would be considered an appropriate one.

Thus it is required that those fundamental topics and concepts in mathematics be identified, from the most elementary upward, which are basic to all views of mathematics and which help unify the body of mathematics. A curriculum that fulfills this function would be considered sound. To be considered efficient, it should be pruned of unnecessary topics and of ideas that would need to be unlearned later.

Another assumption of CSMP is that mathematical content which is sound and appropriate, in the above sense, will also be enjoyable if the student is actively involved in its development. Consequently, CSMP has the goal of designing materials and methods of presentation which will bring students into the learning situation as participants rather than as observers.

In answering question 2, concerning adaptation of curricula to individual students, it must be admitted at the outset that goals set _a priori_ will be in the form of hopes rather than of expected outcomes. It is hoped, for example, that certain learning characteristics in a child can be identified that will influence the content and presentation of a curriculum for this type of child. It would be unreasonable to set this as a goal in view of the paucity of information and the lack of a theory of learning. CSMP has decided that the reasonable course is to select a group of children and, through experimentation, find what content and presentation seem best for these specific children. In the beginning the learning characteristics of these children are simply their demonstrated abilities to understand and the speeds at which they move through the various experimental topics and presentations set for them. Evaluation methods must be developed to decide what content and presentation seem best at a given stage for a given child. Also, a system must be designed to manage the process of experimentation, evaluation, correction, and adaptation. Thus, an expected output of the program is a set of materials, media, etc., that can be studied individually at various paces and which can be adapted to individual learning characteristics when they are identified. The corresponding outcome expected in the behavior of students is a general rise in mathematical literacy and the ability to use mathematics creatively.

CSMP decided to start its program at two distinct levels. For students of ages 8 to 12 the staff has tackled the problem of identifying a basic core of elementary mathematics and has been experimenting, at a 3rd-grade level, with various media for its presentation. In September 1970 this portion of the program will be extended to children of ages 5 to 12, with the development of K-1 materials with a group of children in the SIU Laboratory School; work with 2nd graders will begin in 1971. Meanwhile, development and pilot testing of materials for grades 3-6 will continue as indicated in the CSMP Scope of Development chart in the Appendix. With another group, namely, with students of ages 12-18, the staff is working at a more sophisticated level to find upper limits of content for the secondary school in the framework of an independent study program. So, in this initial phase, the staff is learning methods of "packaging" elementary materials, while studying and teaching more advanced mathematics. A future phase will bring these two aspects of the initial phase together, with the expected output a curriculum and a sequence of "activity packages" for each student from ages 5 to 18. (See Component II.)

The developers of the program believe that, to realize the above goals, CSMP must be discipline-oriented. By this they mean that, while all pedagogical aspects of mathematics education are of deep concern, priority is given to the selection of mathematical content that is sound, relevant, and enjoyable. "We do not want to do a superb job of teaching trivia."[1] The implications of this decision are that the mathematical community must be deeply involved in the program, that there be mathematicians physically in residence, and that mathematicians must guide the program. To date this has been the case and it is a principle of CSMP procedure that every phase of the future development of the program will continue to have a strong involvement of mathematicians. As overseers of this principle, the National Advisory Committee of CSMP will always include a number of prominent mathematicians. (See list of members in Appendix I.)

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[1] Dr. Wade M. Robinson, Executive Director of CEMREL, to the CSMP Steering Committee at its first meeting in St. Louis, May 1967.

STRATEGIES FOR REACHING THE EXPECTED OUTCOMES

The following activities constitute the initial phase of the program: 1) identification and analysis of mathematical content; 2) experiments conducted in packaging materials at the elementary level; 3) experiments in writing and teaching designed to find upper limits of content at the secondary level.

The next section will describe in detail the three components of CSMP:

I. Curriculum Content and Its Analysis
II. Activity Package Program
III. Elements of Mathematics Program.

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II

PROGRAM COMPONENTS

- Component I: Curriculum Content and its Analysis
- Component II: Activity Package Program
- Component III: Elements of Mathematics Program

PROGRAM COMPONENT I

CURRICULUM CONTENT AND ITS ANALYSIS

PROGRAM COMPONENT I

CURRICULUM CONTENT AND ITS ANALYSIS

CSMP is a content-oriented program. Its goal is the development of mathematics curricula that are sound in content and appropriate for the needs of future adults in a changing society. In developing strategies for Component I, the mathematicians on the staff of CSMP looked for answers to the following questions: how to identify the content of elementary (pre-college) mathematics that is basic to all views and all uses of mathematics; how to organize this basic core of principles, concepts, and methods so as to exploit their unifying features.

Some obvious alternatives were immediately rejected. For example, one could survey the current commercial textbooks and, from a statistical analysis of the topics most frequently presented at each grade level, identify a basic core of topics. It is clear that such a procedure in no way reduces the current dissatisfaction with mathematics education. Any other alternative that would restrict the selection of content to conform with any existing curriculum or existing organization of topics was also rejected.

SOURCES OF INFORMATION FOR DECISIONS IN CURRICULUM DEVELOPMENT

A project like CSMP has to be open to any suggestions which might affect its goals, its choices of content, its techniques of presentation. This means that CSMP should have channels of information about trends in mathematics as a developing science, as well as the results of research in mathematics education, both here and abroad.

Such channels of information are provided by several CSMP groups and organizations: 1) National Advisory Committee (NAC), 2) Staff Associates (SA),

3) Consultants, 4) annual international conferences on teaching various mathematical disciplines.

The **advisory committee** gives general views of current mathematics education and reviews broad policy decisions. The **staff associates** and **consultants** keep the staff informed concerning trends in various fields of mathematics and their possible influence on the content of the curriculum. For example, consultants have presented papers on the theory of categories and functors, non-standard analysis, and the mathematics of physical quantities. Staff associates provide expert advice in their fields on a continuing basis. (See pp. 35-37.)

A fruitful channel of information has been opened by a series of CSMP international conferences on the teaching of specific subjects. The first of these was held in Carbondale in March 1969, to investigate the teaching of probability and statistics in the schools. The second, on the teaching of geometry, was held in March 1970. (Appendices VI and VII list titles of papers presented and recommendations.) Future conferences will examine the teaching of algebra (1971), logic and foundations (1972), and analysis (1973).

Communications are kept open with mathematics educators from other countries. Several have already contributed significantly to the CSMP content component through active personal participation in the CSMP international conferences, by occasionally collaborating in the writing of CSMP materials, or by their services as consultants to the CSMP staff, particularly with reference to ongoing research and projects abroad in the field of mathematics education.

CRITERIA FOR CONTENT SELECTION AND CURRICULUM DEVELOPMENT

It was decided that content selection should be unfettered by traditional notions of "what children can do" or "what teachers can teach". It should, instead, be guided by what is important in mathematics, by the uses of mathematics, by what the outcomes of mathematics instruction should be, and by what students actually demonstrate they can do. Content selection follows these general principles:

1. Adequate Coverage. When one looks at all of mathematics from its foundations to its applications, one finds that the subdivisions a hundred years ago, such as algebra, geometry, analysis, etc., are being blurred as development continues. Nevertheless, there are still some distinct areas

in mathematics in which there are differences in content, in problems, applications, methods and styles of thinking. From these various areas one must select content which is basic to all and yet represents typical aspects of each. The application of this principle to the modern blending of disciplines identifies topics which unify concepts in mathematics; such topics are, for instance, sets, relations, and mappings. By this same principle one also identifies certain mathematical activities, tools, and methods which are not necessarily part of particular areas but which are generally needed in all of mathematics, such as algorithms, mathematical language and logic, axiomatization, and heuristics.

2. Viewing Mathematics as both Art and Science. Since mathematics is both an art and a science, the content chosen should illustrate and nourish both of these aspects of mathematical activity. The art of mathematics cannot be programmed: it demands insight, it depends on the ability of a student to make imaginative leaps, to guess intelligently, to reason by analogy, to select the appropriate mathematical instrument.

Any well-rounded mathematical education must help prepare a student to practice the art of mathematics, if only in a modest way. This should not be an objective only for future research mathematicians, but for anyone who hopes to use mathematics in any but the most routine way. The art comes in the passage from a special case in a situation to a generalization of the situation. The process of passing from the general to the particular is the scientific aspect of mathematics.

3. Utility. The recognized usefulness of mathematics in our modern world requires several levels of outcomes for mathematics instruction. At one level, a modern person must be able to compute, to draw quantitative conclusions, to read data, etc. At another level, he must develop a freedom of mind. This requires an attitude that mathematics is not a fixed set of computational formulas from which to draw when solving a problem. It is a freedom of mind and a flexibility that allows one to develop his own methods for solving problems. At still another level, one needs to acquire the ability to think in mathematical terms, to build

mathematical models of real situations, and to reason deductively. Content should be selected in such a way that each student will reach as many of these levels as possible.

ANALYSIS OF CONTENT

Once specific content has been identified according to the above criteria, there must follow an analysis of its place in the total curriculum of the program. At each stage the curriculum content goals must be kept in mind so that, from the beginning of the program, topics and mathematical experiences are chosen which eventually provide the experiences from which the mathematical concept will be abstracted. It is not enough to select topics on the basis of a methodology or on the basis of a short-term goal. Such practice may in fact be harmful if later an idea must be unlearned, or if it results in a narrow or restricted view of a concept that later needs to be understood in a more general way. A continual re-analysis of content should produce a curriculum that does not once more become locked in and fixed as mathematics continues to grow.

Analysis of content may even show that certain so-called "traditional" subjects in elementary school may be preferable to a "modern" treatment. One such example is the teaching of rational numbers. Traditionally, rational numbers are taught in connection with concrete objects being separated into congruent pieces, such as a cake cut in quarters, to represent rational numbers as unions of congruent parts. From a mathematical standpoint, this approach has been criticized, since physical objects do not belong to mathematics. The mathematical approach to rational numbers as equivalence classes of ordered pairs of integers, standard in university algebra textbooks, was then proposed for school mathematics. An analysis of the traditional and the ordered-pair approaches shows interesting results. The development of rational numbers as ordered pairs, surprisingly, can be presented on an elementary school level, but, in doing so, the numbers being developed lose connection with the realities to which they are linked in daily life and science. On the other hand, an analysis of the traditional approach shows that there is really nothing mathematically incorrect in "adding" quantities. A theory of quantities can be developed for which the traditional presentation of rational numbers is a partial model and from which the notion of a rational number in terms of parts of a whole can be made mathematically precise.

For the development of a curriculum this means that when dealing with quantities, such as pieces of cake, sticks of wood, etc., one should first consider the mathematical structure concerning these quantities, and then transform this structure into learning activities for children: sticks of wood can be put together to make longer sticks; sticks can be compared as to whether one is longer than the other; for a given stick, another exists which fits three times onto the given stick; etc. These and many related activities are now mathematically justifiable and mathematically important. They prepare not only for the concept of rational numbers, but also for the many relations the physicist has to observe between numbers and what he calls magnitudes.

SIGNIFICANT OUTCOMES OF CONTENT ANALYSIS

Final decision with respect to content selection for the CSMP curriculum rests with the CSMP Staff Associates and the program director.

Of significant help to the mathematicians are the proceedings of the CSMP international conferences on the teaching of mathematics at the pre-college level.[1]

One of the main functions of the CSMP Staff Associates is the production of a series of "Mathematicians' Outlines for Package Development". These documents are prepared in accord with a set of practical "Guidelines for Writing Outlines"(see Appendix III.A), and are designed to provide mathematical guidance for the staff during the development of the CSMP curriculum and teaching materials. Besides giving a broad view of a particular topic, the mathematicians' outlines describe in detail why certain approaches or points of view should be preferred, and how certain concepts, methods, and tools can be embodied in various activities which may be studied over a two- to three-year period. They point out relations among the various existing outlines, and indicate ways to incorporate different concepts in one and the same situation.

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[1] The proceedings of the 1969 conference on probability and statistics have been published in book form by Almqvist & Wiksell, Stockholm, Sweden. The papers presented at the 1970 conference on geometry will also be published and will be available in 1971.

Members of the CSMP writing staff use these outlines to develop "Activity Sequence Outlines", (see Appendix III.D for sample ASO), which spell out the details of a specific package, including the sequence of activities involved, suggestions for appropriate problems, media, and motivation for the ideas being presented.

The mathematicians' outlines and the activity sequence outlines are intended specifically for the teacher-writers who work with children in experimental classes, who plan daily lessons for them, and later prepare materials in the form of activity packages for other classes. (See Component II for a description.) Experiences in these classes result in feedback to the mathematicians and influence further planning.

An important resource for the work being done by the CSMP staff and staff associates is a documentation center in the CSMP quarters. Headed by a mathematics librarian, the center is gathering an adequate collection of mathematics books and journals, and information pertaining to experimentation in mathematic and mathematics education. The center has developed a new classification system for its holdings and is designing a storage and retrieval system for the activity packages as they are produced, with regard to their content and student characteristics. This resource is provided as a support function of CSMP by CEMREL through its Educational Materials Center.

INVOLVEMENT OF MATHEMATICIANS

The identification of sound and appropriate content for a curriculum is a job for mathematicians with deep concern for the criteria set out above. Fortunately, there are a number of prominent research mathematicians who have turned their attention in the past decade to the problems of mathematics education, and the number is growing.

From the beginning of its existence CSMP has sought the services of many from this group for its advisory committee, its resident staff, and for its staff associate organization. Although it was possible to attract many of these mathematicians to short planning meetings, where they unanimously agreed that CSMP needed long-term involvement of the mathematical community, it was found extremely difficult to recruit a sufficient number of mathematicians for sustained periods of time.

In order to insure sufficient mathematical leadership of the program, CSMP now operates as follows: a group of prominent mathematicians form the CSMP National Advisory Committee. (See Appendix I.) They are kept informed of the program's progress and are asked to contribute their scholarship and experience to CSMP through written comments and through occasional meetings and conferences.

Since there has been a lack of continuity in the tenure of resident mathematicians, and because the help of specialists is indispensable in planning for the program's mathematical direction, CSMP requires the services of a group of staff associates. These are mathematicians who combine competence in mathematics with a deep interest in mathematics education and who agree to spend various periods of time with the program to plan, oversee, and monitor the production of materials in their special fields. At an organizational meeting in May 1969, the staff associates agreed on a structure for coordinating their contributions to the project, and delegated specific assignments in the writing of mathematicians' outlines in accordance with the set of practical guidelines mentioned above (p. 32), in the monitoring of the work of the staff, writing materials, and serving in editorial capacities. Regular meetings of the group of staff associates were planned to review work in progress and to discuss current mathematical questions arising in the writing and planning. The first four of these meetings were held in October and December 1969, and in January and April 1970. The second one was held in conjunction with a meeting of the National Advisory Committee, the third one in San Antonio, Texas, on the occasion of the meeting of the American Mathematical Society. The April meeting was held one week after the International Conference on Geometry. Other meetings are planned for June, October, and December 1970, and January, April, and June 1971. To date, the staff associates include the following: a mathematician whose major role is writing the EM program (see Component III); several algebraists who are writing content outlines and EM books on a variety of topics in algebra, topology, number theory, graph theory, etc.; several analysts who are writing outlines and EM books on measurement, rational numbers, computation, etc.; two probability and statistics experts who are developing initial outlines in probability and writing the probability books in the EM series; a geometer who is writing geometry and linear algebra outlines.

A staff associate has been chosen to act as coordinator of staff associates activities. His major responsibility for this year consists in organizing and expediting the work of the staff associates and in acting as liaison between them and the CSMP staff. He will also be involved in the long-range planning of the program content of CSMP, including the secondary school level and the training of teachers. He is ex officio chairman of the staff associates meetings and conducts CSMP staff seminars.

The staff associates are considered part of the CSMP staff: they receive all staff notices and bulletins and, between meetings, keep in constant touch with the Carbondale staff.

It is necessary to expand the group of staff associates in the next phase of the project to include specialists in numerical analysis and computers. Recruiting of such specialists is currently underway.

One final note should be added to this Content Analysis section of the Basic Program Plan: the April 1970 staff associates meeting was wholly dedicated to laying the groundwork for a master plan of CSMP K-6 content development. Its need is becoming every day more urgent for the reasons spelled out in detail in Appendix IV.

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PROGRAM COMPONENT II

ACTIVITY PACKAGE PROGRAM

PROGRAM COMPONENT II

ACTIVITY PACKAGE PROGRAM

Activity Package Production is the major component of CSMP. This component is concerned with the production of a store of activity packages which will be used as building blocks of the mathematics curriculum for individual students. The actual choice of a package sequence for a particular student will be made with the aid of data furnished by our evaluation experts. In practice, the sifting of the mathematical literature, the collecting of ideas, suggestions, attitudes, and comments from the mathematics community through meetings with experts and interested parties, and the analysis of various efforts (both successful and unsuccessful, in house and out) will yield a large reservoir of possible mathematical content matter for grades K-12. This material is investigated for soundness, desirability, and feasibility (see Component I, pp. 23-37) by the mathematicians and mathematics educators associated with CSMP. After topics have been identified and accepted as generally sound and appropriate, and detailed outlines and rationale for them are developed, there follows the preparation of materials for the teaching of this content to the children of a certain age.

The staff of CSMP faced the following questions with regard to Component II: how to arrange the topics and how to present them so as to realize the outcomes desired.

Several alternatives were considered: 1) Some form of programmed workbook which pupils would study with self-pacing; 2) A series of textbooks, each written with certain learning characteristics in mind; 3) A store of short learning units, here called "activity packages', that could be put together in various sequences to form curricula for individual students. The third alternative was initially chosen in principle for the following reasons: a single continuous set of materials, such as programmed workbooks, might not lend itself to variations in media and might not allow for differences in children other than speed of comprehension; standard textbooks would ordinarily require the usual teacher-centered classroom setting, with resulting reliance on teacher preparation and with the possibility of non-involvement on the part of the students; textbooks, even if they could be directed to specific student characteristics (which does not seem feasible, given our present knowledge about the learning process), would still not meet criteria for individualization of learning.

GENERAL GOALS FOR ACTIVITIES ON THE ELEMENTARY LEVEL

The CSMP Activity Package Program component will ultimately encompass the entire K-12 range. At present, however, because of the actual SIU Laboratory School structure, the CSMP staff situation, and the reasons suggested on pp. 51-52 of this Basic Program Plan, the decision was taken to begin package development at the 3rd-grade level.

1. The children in Grade 3. Since the children in Grade 3 have not, as a rule, been exposed to a curriculum comparable to CSMP, these children must be given basic experiences which, in the full program, may well belong to the K-2 period.

In fact, observations indicate that these third graders -- even the brighter ones -- often do not have the skills one usually expects from them in light of the traditional emphasis on arithmetic skills. They may be able to name numbers up to the hundreds, but they frequently have only an incomplete understanding of the decimal system and know little about adding natural numbers according to their decimal notation. Many of them still add by counting.

Fortunately, third-grade children seem to be still capable of enjoying many of the activities which, in our view, should have come earlier in their mathematics education. Because of this, instead of merely rounding off those skills which 3rd graders are supposed to have from their previous schooling, activities are designed which will involve them in original situations from which mathematical insight and additional skills can grow. On the other hand, they are not treated as absolute beginners, but as much use is made of their actual learning level as possible.

2. Mathematics for young children. The traditional elementary school mathematics curriculum was

heavily oriented towards computation drill, arithmetic, and various kinds of measurement. Overlooked and often neglected were the several ways in which the properties of numbers could be explored and the many ways in which a child could encounter mathematics in his environment, and by which he could be encouraged to become mathematically active. In the activity package approach, some of these ways of meeting and treating mathematics can be made possible, for example, by dealing with sets of objects of different kinds and with their interrelations; by dealing with physical magnitudes (as groundwork for later development of fractions, measure, etc.); by observing geometric patterns and features in commonplace situations; by dealing with operations other than those of traditional arithmetic; by using diagrams, graphs, tables and other tools in organizing data and representing a structure; by highlighting the role of patterns and symmetry in treating collections of data, organizing materials, analyzing systems, etc.; by doing deterministic and random experiments, keeping records of the outcomes and developing an adequate probabilistic mode of reasoning. It is not difficult to see that a rich variety of experimentation, observation, visualization, organization, analysis, computation, and discussion can be opened to the child. We hope that by such an exposure every child will get an optimal chance to find his own best relation to mathematics and that an adequate basis will be laid for his eventual further studies in the various branches of mathematics.

PROVISION FOR INDIVIDUALIZATION

Provision for self-pacing is a characteristic of nearly all programs that incorporate individualization. It was proposed by CSMP to go beyond self-pacing to some form of tailoring of curriculum materials to the individual needs of students. The concept of tailoring to individual needs suggested some sort of classification of students into relatively homogeneous groups, so that there would be enough similarity within groups to permit the tailoring of packages to suit the idiosyncrasies of the group concerned. Guilford's factor analytic studies[1] have identified numerous dimensions that are apparently related to mathematics learning and, hence, the identification of clearly defined groups of students seems theoretically feasible. If, on the other hand, mathematical learning, especially in younger children, depends on a single factor, or if, as Piaget[2] suggests, all students proceed through the same developmental phases of reasoning, then the classification of students would be along a single continuum and the concept of tailoring would be rather simple.

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[1] Guilford, J. P. et al. "The Structure of Intellect." Psychological Bulletin, LIII, (July, 1956), pp. 267-93.

The theoretical differences in individualization have been left unresolved, because there are very practical reasons for adopting another concept of individualization. From a financial point of view, the problem is presently too costly to tackle. One must also keep in mind that measurement of different abilities in young children at this point in history is not sufficiently discriminating to permit precise allocation of students to homogeneous groups. Another more compelling reason is that, during the design and development cycles, CSMP deals with a small group of students, and the possibility of identifying typical representatives of the required student categories is very remote. Even if good representatives were identified, the effects of tailoring to meet a given goal would be almost impossible to distinguish from measurement error. While not rejecting the possibility of tailoring in the future, CSMP's decision to abandon it for the present is consistent with a decision made by Glaser[3] in connection with his development of Individually Prescribed Instruction, at which time he concluded that "proof will have to be forthcoming that the selection and devising of instructional materials does indeed interact with student differences, so that their achievement in seeking a given educational goal is significantly better than if an average best method were used."

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[2] Inhelder, Barbel, and Piaget, Jean. The Growth of Logical Thinking from Childhood to Adolescence. (Translated by Anne Parsons and Stanley Milgram.) New York: Basic Books, 1958, pp. xxvi, 356.

The approach to individualization beyond self-pacing that has been adopted by CSMP is that of providing a broader array of topics than any one student can possibly master, and the multiplicity of topic sequences that the student's ability and interest will permit. One further dimension to individualization being explored within CSMP is complexity of treatment. For example, any given content topic may be treated in one package at a rather unsophisticated level that is adequate as a foundation for some types of further work in mathematics, and that same topic may be treated in a parallel package to a much greater depth that will provide a much broader basis for continuing work in mathematics.

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[3] Glaser, Robert. Working Paper 1: "The New Pedagogy". Learning Research and Development Center, University of Pittsburgh, 1965, p. 4.

SURVEY OF SOME ATTEMPTED CSMP ORGANIZATION PLANS FOR PRODUCTION OF MATERIALS

With the alternative chosen to prepare a store of activity packages, ten teacher-writers of CSMP were assigned to Component II. They started by producing several prototype packages based on isolated parts of content in the elementary program, complete with varieties of media such as video tapes, audio tapes, games, written materials, etc. Through this experience the staff learned to use various instructional media and, at the same time, produced a few packages for the envisaged "store" of packages.

The initial organization of package production consisted simply of four teams of two or three staff members, with each team working on an outline of a mathematical topic (developed as described below), deciding on a level of sophistication, and producing media and materials that they considered suitable to present the topic as a learning unit. The original topics chosen tended to be rather broad, so that some of the prototype packages, when tried with a few children, took about a month to complete.

Experience with the initial organization exposed several flaws in it. First, the working hypothesis that all staff members were equally effective in each aspect of package production was rejected. Second, the practice of different teams producing packages for different topics caused problems of consistency of language from one package to another, and the more serious problem that the prerequisites for one package were not being supplied in another. Third, the prototype packages did not have continuity of development from one package to another, and the possibility of producing sequences of related packages posed difficult problems of liaison between teams.

After six months of this necessary experimentation with organization and development, a different mode of operation was suggested. The new plan was designed to take into account the strength of each staff member and called for production of one continuous sequence of packages, with each staff member serving mainly as a specialist in one aspect of production. For example, certain people would have the major responsibility for organizing the topics and outlining the activities, others for the most part would prepare activities and media, while others would be mainly concerned with attempting to identify ways to adapt the resulting packages to different kinds of pupils.

The staff was now ready to go into actual production. The elementary grades of the Laboratory School of Southern Illinois University were used as a source of pupils for tryouts. At this point the third-grade level was chosen as the place to begin the experimental package program because it was felt that the mathematical content would be basic, yet there would not be the problems of psychological adjustment to school found at earlier grades. Development of K-2 level materials will begin in September 1970.

In the fall of 1968 the 28 pupils in the third grade at the University Laboratory School were separated into "Class A", "Class B", and "Class C". Pupils with somewhat the same reading ability and general high achievement in school were placed in A and B. The rest were put in C. The students were brought to the CSMP headquarters (where the necessary equipment is located) one hour each day.

The plan was to let Class A begin by studying a prototype package that was considered suitable for this age.

Then, on the basis of this experience, the package would be revised and improved before being used again with Class B. At the same time it would be adapted for use with Class C. Shortly after the start of this operation it was realized that two incorrect assumptions had been made. In the first place, it had been assumed that materials could be prepared *a priori* for children of a given age. Furthermore, it had been assumed that one could predict in advance the extent of the ability and willingness of young children to concentrate on one topic.

Again, the mode of operation was reappraised, and the question of what constitutes an activity package was reopened, this time to be left open. It was decided that the division of labor in the organization should not be quite so rigid, and that the decisions concerning the presentation of topics should be made entirely on the basis of experimentation in Class A. Furthermore, no topic should be continued for an extended period of time (as in the prototype packages) without abbreviating it and blending it with other related topics.

The 1968-69 organization of package production that evolved (see Appendix II.A for a flowchart) was based on an empirical and inductive approach to curriculum development. It assumed that one could not predict how or what a given group of children could learn, and proceeded to develop materials for these children by engaging them in short mathematical activities, observing the results, making tentative decisions on the basis of the observations, preparing materials to implement the decisions, trying the materials with a comparable group of children, again revising the materials on the basis of the trials, adapting the materials for other children, trying these materials, revising them, and so on, until one or more versions were ready for more general use.

This process of cyclical trial with third-grade pupils yielded sequences of short activities. These materials constitute about one-half of an integrated program that eventually will cover a full mathematical curriculum for students of ages 5 to 18. The concept of an activity package is now a changing one. Whatever sequence of short activities that seems to have some natural unity or that seems to have a natural point of termination is considered to be an activity package. At this point some sort of evaluation is made to determine, on one hand, the effectiveness of the package and, on the other hand, the student's readiness to do another package.

THE PRODUCTION OF AN ACTIVITY PACKAGE

Appendix II.B gives a diagrammatic summary and a detailed description of the activity package production organization which CSMP has currently adopted and which will be now briefly described.

A mathematician, selected from among the Staff Associates, is charged with preparing what is known as a Mathematician's Outline for Package Development. (Cf. App. III.B for a sample mathematician's outline on The Integers.) This document is prepared in accord with a set of practical "Guidelines" (cf. App. III.A) drawn up to ensure that the outline will indeed provide the needed mathematical guidance for the writing staff during the development of the materials. Besides giving a broad view of a particular topic, the mathematician's outline describes in detail why certain approaches or points of view should be preferred, and how certain concepts, methods, and tests may be embodied in the actual activities which the students will cover over a two- to three-year period.

Members of the CSMP writing staff use these mathematicians' outlines to develop Activity Sequence Outlines which spell out the details of a specific package, including the sequence of activities involved, as well as suggestions for appropriate problems, media, and motivation for the ideas being presented. (App. III.C gives a sample activity sequence outline on Motion Geometry.)

A preliminary version of an activity is produced, through actual teacher-student contact, by "Team A"[1] working with a development group of students selected from the appropriate level at a local school. A first version of the various activities is then tried out with a full class of school children.

The first version of the activities then undergoes pilot testing in a setting not completely under CSMP control, viz., in a dozen classes using available public school teachers who are asked to participate in regular in-service training sessions throughout the school year and during the summer. The pilot test is designed to yield valuable "formative" evaluative information which is fed back to the writing staff to produce a revised version of the activity. (Cf. App. III.D for a sample activity in Motion Geometry.)

The pilot test experience is also indispensable in the devising of suitable classroom management models for subsequent field trials and wider diffusion through interested school systems.

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[1] "Team A" was originally chosen as the name for the group of teacher-writers working with Class A students (cf. p. 51). The name is being kept even though Class B and C have been discontinued.

ACTIVITY PACKAGES CURRENTLY IN PRODUCTION

As is apparent from the above description, an activity package consists essentially of a flexible number of "activities" designed to develop a mathematical concept or skill in a child at a given level of development. These activities make use of a wide variety of media: programmed presentations of a Socratic nature with immediate built-in feedback, video and audio tapes, games, and a number of manipulative devices.

The CSMP staff has already produced about 170 third grade activities, presently grouped under topics of relevance to 8- or 9-year-old students. The eleven packages are: Addition (31 activities); Geometry (24 activities); Length I (3 activities); Introduction to Multiplication (36 activities); Names (7 activities); Number Line (5 activities); Probability (12 activities); Positional Notation (17 activities); Repeated Addition (18 activities); Residue Class Systems (17 activities); Subtraction (26 activities).

In practice, a student will be exposed to a "preferred sequence" of activities from each of the packages on hand.

Both package titles and the number of activities are constantly changing in this essentially flexible and growing program. Packages are currently being planned and developed (with a group of 4th graders) in the areas of graph theory, rational numbers, measurement, operational systems, number theory, topology, space geometry, integers, relations, and functions.

1969-70 PILOT TEST OF CSMP MATERIALS

In March 1969, after the first few packages had been developed and initially revised, some of them were tried out for four weeks in a third-grade Carbondale classroom. This was the first time that package material had been used in a regular classroom setting. Although several problems were encountered, enough was learned from the tryout to plan a limited pilot test in Carbondale for the 1969-70 school year.

Two questions had to be answered with some assurance before the project could continue to produce new materials with any confidence: 1) Could children learn adequately from the packages? 2) Could a teacher, possibly assisted by a teacher-aide, function effectively and handle all the tasks demanded by the program?

Other issues which were to be investigated in a quasi-natural setting involved teacher training, equipment, storage and access of materials in a classroom, a system that would help the teacher "keep the books" and make sequencing decisions for the students, ways of diagnosing needs and of developing remediation procedures.

The Carbondale public schools cooperated with CSMP by assigning three teachers, with their classes, to use the program for a year. One class was assigned to the regular teacher assisted by a teacher-aide; a second class was entrusted to a CSMP teacher (taking the role of the master teacher) assisted by the regular teacher; a third class was assigned to a CSMP teacher (master teacher) by himself.

The classes were comprised of students from approximately the top 60% of the population (IQ), since the development group of students available to CSMP was in this range. A few students outside this range and with below-average reading ability were added to assist in the development of strategies to meet their needs. The teachers were trained by CSMP personnel during a 2-week period in the summer and through an in-service program during the year.

Although the pilot study could be viewed as an experiment, none of the rigor associated with field testing and comparative evaluation was implemented. Since this was the first real trial of CSMP materials, many procedures were being initiated for the first time, some on an emergency basis; many of these were to be discarded for various reasons, and many decisions were based on direct observation and teacher suggestion rather than on empirical evidence.

The trials were far from complete (a full report will be issued in October), but some tentative conclusions can be drawn:

-- Students like the program and prefer it to the traditional one;

-- Teachers like it, are able to handle it (in some cases, unassisted), and would like to continue using it;

-- The program, as it now stands, does not adequately serve poorer students with reading problems.

-- The cost of the program is rather high for most potential consumers to bear.

The last two items are presently a matter of concern and effort within CSMP. Various alternatives are being investigated and tested in order to reduce the cost.

Three examples may serve to illustrate this effort:

1. Most activities are in the form of self-instructional booklets, which are disposed of when a student has finished the activity and, if necessary, has had it checked. Some form of re-usable activity, which could be used by several students in the same class at different times, or by several classes, and possibly over a period of years, might extend the life of an activity through perhaps 25-50 uses. Lamination of activities with surface coating, use of erasable plastic "write on" pages and separate disposable answer booklets are being investigated.
2. Television is a rather expensive and inflexible tool in the package program. Even with a single video player and two or three monitors the expense is high. The use of cartridge video-tapes which can be inserted into TV monitors may reduce the cost somewhat and increase flexibility by allowing several different tapes to be viewed at the same time. This improvement depends on the speed and consumer cost of the technological improvement. It may be that, even with the most economical and flexible method of video presentation, the instructional gain over other methods of presentation will not justify the added expense.

3. During the 1969-70 pilot study, CSMP personnel assisted the regular teachers in many aspects of the program. A system of activity selection, sequencing, and record keeping is being developed which can be used by the teacher (or her aide) without outside assistance. The use of a teacher aide, which is a costly item, may be only marginally advantageous, especially where students are above average in ability.

The decision to produce a single set of packages, covering a wide array of mathematical topics from which a program could be selected for a particular student according to minimum goals, interest, rate and past record of achievement has meant a rethinking of the writing of materials. Where the content is considered mandatory for all students, such as basic algorithmic skills, the activities must be effective for slower students and not too slow or boring for the more capable.

The rewriting of some high priority packages in this way involves such technical devices as clearer page formats and more extensive use of audiotapes, as well as an effort to lower the prerequisite reading ability through the use of shorter sentences, simpler vocabulary, and more efficient means of checking the work (by teacher and by student).

Where the activities are less important, in terms of minimum goals, the activities can be written in a more interesting way for better students and involve more rapid development of content. How successful these efforts will be remains to be seen. The use of peer tutoring is also under consideration.

Possibly all students may not be ready, for a variety of reasons, to begin to work in the semi-independent way required by the program at the beginning of grade three. These students may prefer and benefit more by learning from the teacher in a more traditional, group-centered approach until they are ready to begin working individually.

Beginning in September 1970, a pilot study will be carried out involving all twelve of the third grades of the Carbondale Elementary School System. In one school, with four third grades, two classes will receive the traditional grade-three program and the other two will receive the CSMP program; all will be taught by the regular teachers. One of each pair of classes will have a teacher aide; the other will not. Thus, an initial estimate can be made of the effects of the CSMP program as contrasted to a fairly typical, traditional one, and of the effects of the addition of a teacher aide to a classroom. Students will be randomly assigned to the four classes; but the teacher effect, the small sample, the selection of only one from a variety of traditional programs, and the fact that the CSMP program has reached only an intermediate degree of stability, will tend to decrease the generalizability of the results. Nevertheless, gross differences in a variety of criteria will begin to indicate the kind of summative data which can be expected in later, better controlled studies.

A second school, with four third-grade classes, will use, as a master teacher, a present third-grade teacher who will be responsible for all four classes. In one of these classes the teacher will teach, as a group, those students who do not make satisfactory progress individually, until such time as they are able to do so. The number of students in this group will depend on the quality of the students: there might be as few as two or three, or as many as nine or ten. This school will also provide for a test of an internship type of training, where the regular teacher will observe and learn right in the classroom the system of instruction, the content of the activities, and how to deal with various problems which arise, and will then gradually assume the responsibility of managing the class.

Two other schools, each with two third-grade classes, will study CSMP materials. In all classes the ability of the slower children to work at an acceptable rate of speed and achievement with the revised materials will be of paramount importance and will undergo careful scrutiny.

It must be finally noted that pilot testing of CSMP materials outside a very limited radius is out of the question under present funding conditions. Every effort is being made to ensure that adequate training is available for the teachers required in the local pilot test classes.

While no wider teacher training can be started at present, the developers of the CSMP materials are deeply concerned, and feel responsible for, the ultimate success and adequacy of such indispensable training: they will develop, therefore, a set of specifications and guidelines for the training of the teachers that will be needed in the field trials and diffusion stages of the program.

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PROGRAM COMPONENT III

ELEMENTS OF MATHEMATICS PROGRAM

PROGRAM COMPONENT III

ELEMENTS OF MATHEMATICS PROGRAM

The CSMP activities are presently concentrated at two levels. At one level there is an effort to find, through experimentation, ways to identify content and use various media to construct a sound curriculum for all students of ages 5 to 12. At the same time, at another level, CSMP is addressing itself to some related problems:

a. What is really an upper limit of content for students of ages 12 to 18 that could then be used by CSMP as one curriculum for capable students?

b. What can be done to exploit the mathematical potential of the capable students in our schools?

CSMP made a decision to develop a particular curriculum and a particular approach for capable students as an experiment designed to answer these questions. Among the various alternatives for content and approach, CSMP chose one that they believe has not been tried before. It is one in which, after some heuristic background, the initial emphasis is on logic and set theory, which are used thereafter as tools in studying the standard mathematical systems as well as in mathematical model building.

One would expect that any particular choice of content and approach would lead to controversy in the mathematics community. Some prominent mathematicians, including some consultants and members of the first CSMP Steering Committee, objected vigorously to the choice made for the EM curriculum. They argued that it was overly formal, too rigidly tied to logic, and overbalanced in algebra and finite processes. They say, in effect, that they can predict the result of such an experiment: it will reduce the mathematical creativity of the students involved.

Others in the mathematical community, whether or not the approach of EM is to their taste, welcome such a curriculum effort as a needed experiment. They see it as the first attempt that has been made to determine the educational effects of a logic-oriented program. If carried to its conclusion and if carefully evaluated, such an experiment might reveal that capable high school students have greater mathematical potential than was realized. As a by-product, it could provide evidence to help resolve some of the controversy over the role of logic in mathematics education.

CSMP feels that the EM experiment is well worth carrying out, in spite of the objections cited above. The following is not only a description of, and a rationale for, the program, but also a defense of the choice of its content and approach.

OBJECTIVES OF THE ELEMENTS OF MATHEMATICS PROGRAM

For children of about 12 to 18 years old, with good reading ability and generally in the upper 15 to 20 percent of the school population, the EM curriculum materials are being developed to carry the main burden of instruction, to be used by a student at his own pace, and with the teacher playing a new role in instruction. We want the student to develop certain general understandings, attitudes, and behaviors, as indicated in the instructional aims listed on pp. 11-13. More specific aims will be dealt with in the discussion of existing and planned materials that will be covered in some 15 books.

On the basis of our decision to have a strong emphasis on logic and formal methods we single out, in addition to the general aims of instruction, the following aims for capable students:

1. We want the student to know something about the proper role of logic; to know what a rigorous proof is, and to see how such knowledge assures that one knows what his presuppositions are; to understand that to grasp a proof is to be concerned with the main line of the mathematical argument and the reason for its choice rather than with the logical machinery. We want the student to distinguish between belief and provability; to be able to construct correct proofs, and to present his proofs in a style appropriate to the occasion, particularly in a style appropriate to his audience.
2. We want the student to see how parts of mathematics can be organized axiomatically; to understand the

relation between a theory and an interpretation of it; to know what is meant by a model of a theory; to know what a mathematical model is. We want him to have experience in developing his own theories and models. In this connection, we want our students to understand the difference between the development of a theory and its representation in an axiomatic deductive manner.

3. We want the student to learn how to read good mathematical language; how to read sophisticated notation; how to create and use notation appropriate to the problem at hand. We want him to see the following position as reasonable and to subscribe to it: when he writes or talks mathematics he must make every effort to say precisely what he means in language appropriate to his audience, while when he reads or listens he must strive to understand what is meant, however it may be presented.

4. We want the student to understand what it means to say that two statements or names have the same form: to understand that logically equivalent statements, as well as as names for the same object, may have different meanings, owing to their different forms. We want him to see that appropriateness

of form is a matter of circumstance of use rather than a matter of arbitrary canons.

ALTERNATIVES FOR THE EM PROGRAM

One could possibly design several different programs to satisfy the objectives listed above. For example, one such program might be based on an intuitive approach to mathematics in which formalization is minimal, methods of proof are learned by following examples, and content arises out of real situations. Another alternative might be developed through applications to physical and behavioral sciences, with no formal theories, and with major stress on computing, estimating, and experimenting. Another program would be a formal development of the standard mathematical systems, and some applications of these systems, with logic used implicitly. Still another alternative would be to begin with an explicit study of logic, language, and sets, and to use these consistently in organizing and studying mathematical structures and some of their applications.

The latter of these alternatives was chosen for various reasons:

Experience in teaching school mathematics has made it clear that the lack of sufficient emphasis on logic-related questions (such as the use of variables, implication and inference schemes, especially when dealing with replacement) has caused mathematics to remain a puzzling mystery for many students who might otherwise have delved more deeply into it.

The haphazard osmosis-type method by which matters of logic are occasionally treated leaves completely open and uncharted for many students the role of logic in mathematics, making it impossible or very difficult for them to understand the difference between content-bound and content-independent reasoning.

The explicit concern with logic, formal language, and methodological questions might well have a definite transfer effect on the role of language and reasoning in other fields of the student's curriculum.

A separate course in logic does not exclude or belittle previous or simultaneous work with intuitive or experimental aspects of mathematics. Our approach, however, keeps the two concepts separate and distinct for a while so that, when they eventually merge again, as they very soon do, the student knows which is which at every step of his work.

Our experience with 13- and 14-year-old students of high verbal ability indicates that they are ready and eager to study formal techniques in a separate course. They seem to appreciate the type of security and the possibility of self control that such a course provides.

Nor should one ignore the expanding market for formal logic: much of the recent growth of interest in, and support for, mathematics is directly attributable to the dawning of the computerized age. A good understanding of logic and elementary model theory is important to the rapidly increasing number of computer-oriented students coming out of the schools. Logic, abstract algebra, finite combinatorial analysis, and similar discrete and rather formal topics, are becoming indispensable tools for the applied scientist. There is also a growing use of logic and set theory within mathematics itself. Recent developments, such as non-standard analysis and model theory, make it important for students to have some understanding of logic, axiomatics, and formal theories in order to comprehend what is happening on some of the frontiers of mathematical research. The alternative chosen for the EM program, with its initial emphasis on logic, seems compatible with these trends.

It is also felt that the individualization of instruction might best be met by this alternative, since the material provided for study will be sufficiently explicit to enable students to proceed to a large extent independently. (The roles of individualization and language in EM are discussed in the following sections.)

In choosing an alternative for its current EM program, CSMP is not ignoring other possibilities. It is anticipated that when the activity package production is at the stage where packages are being developed for students of ages 12 or 13, CSMP will search out and identify other approaches in order to prepare materials suitable for all students of ages 12-18.

INDIVIDUALIZATION IN THE EM PROGRAM

One way to individualize is to provide a one-to-one correspondence between teachers and students; another way is to group children into "tracks", each track with its own program. For the past 15 years, an easy administrative solution for dealing with above-average students has been to identify capable 7th graders and "accelerate" them by shifting the traditional curriculum down a year — 9th grade algebra in the 8th grade, 10th grade geometry in the 9th grade, etc., then dragging freshman calculus down into the 12th grade. The results have been often quite disappointing. Professor Howard Fehr of Teachers

College, Columbia University, reports of a doctoral dissertation based on an 11-year study of a large metropolitan school supporting this contention.[1] The procedure may be questioned on several grounds. If a child can accelerate through the present 7th and 8th grade material, why should he not accelerate through the rest? Surely he experiences no real acceleration after he has been locked into the standard program, even though he does get it a year early. Secondly, is it sound to use material written for the broad spectrum of students for the most capable subgroup? Does teaching it a year early make it more suitable? To supplement with enrichment material is not enough. If the material is not "just more of the same", the student increases his mathematical competence, widens his horizons, and finds the traditional material even less adequate. The notion of enrichment has a fatal flaw when viewed as a means of spreading out the curriculum horizontally for the more capable student so that he will remain in step with the basic vertical program designed for all students. Instead of remaining in step, from this wider base, the student is more likely to make both higher and more frequent upward leaps.

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[1] Ruth Lefkowitz, The Advanced Placement Program in a Large New York City High School, Teachers College, Columbia University, 1966.

With even partial individualization of instruction students' rates of development vary increasingly as the program continues. An apparently homogeneous group at age 12-13 separates so fast that, within one year, either subtracks are needed, or one must be willing to exert pressure on the top and bottom of the group to keep it together.

If n students in a track are split into three subtracks, the teaching staff must be increased, or the available teachers must learn to work with three groups of $\frac{n}{3}$ students doing not only three different things at the same time, but doing them at different mathematical levels. In such a case, on the average, each student can get only one-third as much of the teacher's time as compared with the case of one track. Inevitably this means that the students will be doing more work independent of traditional teacher attention, while the teacher will be dealing with a wider spectrum of mathematics.

This suggests the advisability of changing the teacher-student relation by providing other sources for the bulk of initial instruction, thereby freeing the teacher to deal individually with student development.

Another phenomenon of individualized instruction is the huge amount of material needed to accommodate the better student. The well-motivated student who becomes accustomed to materials written for his independent use at his own speed may set himself a very fast pace indeed. Should he run out of material at the 11th-grade level, for example, he may experience difficulty and loss of momentum when trying to switch to traditional algebra texts. They will not be like the material which he used to arrive at this point, since they are written for the broad spectrum of 11th-grade students, and are designed to be taught by a teacher. To prepare enough instructional material to deal with the individual development of the better student is a large task; it is a truly formidable undertaking to add to that task the production of materials for the individual instruction of all students.

In view of the foregoing remarks, it stands to reason that a first attempt at dealing with these problems should be directed at a fairly homogeneous segment of the school population, and that initially there should be little effort at branching.

There are good reasons for taking an upper track group for this part of the CSMP experiment:

1. Projects such as SMSG and UICSM have produced much fine instructional material in recent years and,

while it is often used with the best students, it is not generally written expressly for them nor to be used without classroom instruction. Coherent, articulated sequences of material for upper-track students are less available than materials for most other segments of the school population. Little is known about how far and at what rate the upper 20% can go with materials designed expressly for them.

2. With the reduced amount of classroom teaching in the CSMP experiment, the bulk of initial instruction is to be carried by the materials. The vehicle with which students are most familiar is written English and the symbolic language of mathematics. To enjoy freedom from other constraints, choosing to prepare materials for students of high verbal ability is a reasonable decision. Such materials may be adapted later to other student characteristics, as experience is gained in using other media in individualized instruction.

3. It is necessary for the experiment that there be long-term participation by students. The sequential nature of mathematics and the major differences

between the EM program and standard programs imply the desirability of keeping transfers into and out of the program to a minimum. Choosing well-motivated, top-track students is one way to retain the integrity of the experimental group. Past experience in the program is affirmative on this point.

4. Of paramount concern to responsible innovators is the welfare of the children involved. In the light of this concern, they must look critically at what they do and respond to criticism both from within and from without the profession. Parental concerns must be their concerns also. The "new mathematics" in the elementary and high schools has proved to be alien to many parents. Many of them feel the need for the reassurance of normative test results that their children will be able to compete adequately with other children pursuing a standard curriculum, and that their college-bound children will not be penalized by pursuing a non-standard curriculum. The usual normative test presupposes a standard curriculum and may have little value as a measure of a non-standard curriculum. Still, top-track students in a semi-individualized program generally stay well

ahead of the norms for their ages. Experience to date in this program bears out the point. While the matter of parental concern is important in the laboratory setup, where student participation is voluntary and communication with parents is relatively easy, the concern becomes more a factor for field testing in public schools. Since new programs are generally required to do as well with traditional (and hence familiar) topics as do the traditional programs, working with top-track students entails fewer constraints than would be the case with most other possible selections.

5. Experience with the upper 15 to 20 per cent will give some idea of the scope of the planning effort required for a program to deal with all students.

THE ROLE OF LANGUAGE IN THE EM MATERIALS

The materials are designed to carry the main burden of instruction for students in the EM program. That is, modification of behavior and extension of capabilities is to be generally initiated by interaction with the materials rather than by interaction of student and teacher. Naturally, there will be interaction between student, teacher, and materials, but in a manner different from that found in traditional classroom instruction. The role of the teacher in this process will be considered later.

The medium of communication in the EM program is written language, which will be interpreted here to mean what can normally be presented on a page, and includes such things as cognitive maps or sketches, tables, and other displays, as well as natural English and symbolic languages. The materials must not only present the substantive content, but must also stimulate in the student activities designed to modify his behavior, such as problem solving, problem construction, working with other students or with teachers, making conjectures, generalizing, abstracting, particularizing, building theories, interpreting theories, etc.

The role of language is important in the general scheme of learning mathematics, and particularly so in a program like CSMP in which students are expected to do a great deal of independent work. Little research has been done on the role of language in mathematics education. It is surprising that this should be so, for, while it is known that much mathematics can be learned through manipulation of material models, the subject remains heavily dependent on language.

The symbolic language of mathematics is not natural English: it is precise, economical, and limited, whereas English is ambiguous, redundant, and rich. Normally the language used to express school mathematics is a mixture of mathematical symbolism and borrowed English. While the student is not required to be articulate about it, he must continually sort out the metalanguage from the object-language, that is, he has to decide of a sentence whether it is a sentence of mathematics or a sentence used to talk about mathematics. The problem is acute when a sentence contains both languages. Almost any discussion of polynomials in a standard high-school textbook will illustrate the difficulties. Clearly, many students find their way through these difficulties, but it is reasonable to hope that improvements in language may be reflected in better progress for more students. It is certainly convenient and practical in mathematics to use certain names ambiguously, but it is not at all evident that it is the best pedagogy to follow these practices in the initial instruction. Pioneering projects in recent years have experimented in dealing with these matters.

It is part of CSMP's plan to make the distinction between object-language and metalanguage more definite in the writing. This is to be achieved partly by making the symbolic object-language more powerful and adequate, and partly by systematic use of well-known devices for keeping languages separate in the writing.

In order to set up an object-language for mathematics, one is forced to employ a metalanguage with which to describe the object-language. This is necessary, however informally one does it, whether it is done orally or in writing. The metalanguage is essentially English, with a small set of added symbols. The object-language is formal; the metalanguage informal.

We have ample evidence that students quickly learn to read and write the formal symbolic language. It is a sparse language compared to English, and cannot compete with English in capacity to deal with subtelties of expression, but it has advantages over English for mathematical purposes. Consider, for example, the following common statements about set equality:

i) Two sets are equal if and only if they have the same members.

This inevitably carries the notion that two different things can be equal, which is an impossibility if "two" and "equal" are to have their usual mathematical meaning. A little injection of symbolism improves matters:

ii) Sets A and B are equal if and only if A and B have the same members.

Since the convention in mathematics is that using different variables "A" and "B" does not commit one to "$A \neq B$", we have been able to get rid of the troublesome "two"; but the plural form remains and still suggests an incorrect notion of equality. The symbolic form is:

iii) $$A = B \Leftrightarrow (\forall x)\ [x \in A \Leftrightarrow x \in B].$$

For those who can read it, this says exactly what one wants to say, with no undesirable connotations. A symbolic sentence like this looks formidable to one not familiar with the language; but this sentence and much more complex ones are not more formidable to a student who knows the language than is Japanese to a Japanese, or Russian to a Russian.

The basic symbolic language in the EM series is that of a first-order predicate calculus with equality, the language of set theory, and, of course, the standard symbols of mathematics. Private notation and words are kept to a minimum, and such nonce words or symbols as are needed are plainly labelled as such. Symbolism is not introduced until there is need for its continued use. No abstraction is given a name (symbolic or otherwise) if only one extension of it is in question. Indeed, extensions are always dealt with before a name is assigned.

The careful distinction between object-language and metalanguage, and the use of an object-language capable of expressing mathematics without borrowed English has not, so far as we know, been attempted to the extent that we have in mind for children at the age levels with which we are working. The whole EM program is experimental, and this distinction of language is one facet of the experiment. There is not much research evidence relevant to this matter, and we feel that there must be a sizable sequence of material written in this vein before one can hope to investigate this aspect of the role of language in mathematics education. For example, if one wanted to investigate how English students could use French as a tool for learning history, one would not investigate the behavior of first-year students of French. Modern mathematics is expressed in a highly efficient, flexible, elliptic language made up of symbols, English, and technical words. We want our students to learn how to employ this language, as well as the dialects of various users of mathematics. But it is not obvious that it is the best instructional procedure to use this language from the outset; indeed, we suspect otherwise. Language is not elliptic for the reader who does not know what has been omitted; it is just obscure. What is an abbreviation to the reader who knows what has been abbreviated is just an unfamiliar word or phrase to the reader who does not. When a new topic is presented in the EM materials, after first dealing with preliminary heuristics, one begins to deal with definitions, proofs, generalizations, etc., in a language that is as explicit and precise as one can make it. As the topic develops and students become familiar with the concepts and processes involved, abbreviation and ellipsis are increasingly employed, both in language and process. Students are encouraged to use their own ad hoc devices with the agreement that they must be able to describe them accurately when communicating with other students and teachers.

By the time students make substantial use of standard text material, they will have had considerable experience with development of technical language, with reasonably sophisticated notations, and with the use and abuse of abbreviation. Furthermore, at that stage, they will be reading mathematical literature which, though abbreviated and mathematically colloquial, is more accurately written than much currently available school mathematics material. Herstein's Topics in Algebra is a good example.

One aim of instruction is to develop in the student, as quickly as possible, confidence in the accuracy and correctness of his work. If a problem has numbers as solutions, the student is encouraged to find all those numbers, if possible, to prove that they are all correct and that there are no other solutions. If the numbers are interpretable as measures of quantities, he learns to check against the interpretation of the given data. He learns to be aware of the actual assumptions on which his work is based. He works on problems whose solution sets are empty, as well as on those whose solution sets are infinite. He is encouraged to estimate where this is appropriate. While many examples are given, the associated problems generally extend the examples. He is not supplied with an answer key. Occasionally answers are given but, generally, he must depend on his own resources for immediate satisfaction, or must consult another student or his teacher.

Directions for exercises and problems are as precise as we can make them, since we fail to see what is gained by making the student guess about such matters. Indeed, making the student guess what might please some authority is opposite to the attitude we wish him to acquire. Directions may range from "Investigate the following situation...." to asking for an answer in some particular form.

There is no canonical form for answers to problems. The student is encouraged to regard correctness of form as dependent on what use is to be made of the result. These students are expected to recognize that $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, but they are not expected, on getting " $\frac{1}{\sqrt{2}}$ " to salivate like Pavlov's dog and change the answer to " $\frac{\sqrt{2}}{2}$ ". If the answer is to a problem clearly designed as a technical exercise, then any form of the answer that satisfies the data and the directions is correct. We would not, for example, give the direction:

Find the solution set of

(1) $$6x^2 + x - 2 = 0 \wedge x \in R,$$

unless we were prepared to accept as an answer

(2) $$\{x \mid x \in R \wedge 6x^2 + x - 2 = 0\}.$$

Of course, we would also accept as an answer

(3) $$\{\tfrac{1}{2}, {}^{-}\tfrac{2}{3}\}.$$

We do not want the student to feel that he must guess which answer would best satisfy some authority. Either answer expresses the solution set, and for some purposes (2) is preferable. We do not want the student to view "$\{x \mid x \in R \wedge 6x^2 + x - 2 = 0\}$" as a call to action; it is simply a name. It should therefore be left alone, unless something one wants to do about the thing named requires that the form of the name be changed.

It is difficult to use in a written exposition the discovery techniques that one readily employs in face-to-face teaching. Still, we believe that the EM materials, in both exposition and problems, have been able to generate such activity early in the program.

PRODUCTION OF EM MATERIALS

The broad structure of the EM program is developed in meetings that include representation from all components of CSMP. More detailed structuring is the responsibility of the writing team, which includes the program director, staff associates, an assistant director, and some teacher-writers.

The initial organization, the exposition, and the exercises in any one book are largely the work of one writer.

The initial manuscript is read by members of the writing group who comment on it in detail and provide additional problems. The manuscript is revised as a result of these comments. The first trial of material is conducted with a group of students under the guidance of an experienced teacher who is generally a member of the writing team.

In accord with the initial trial, the manuscript is further revised and put into book form for trial with students who will pace themselves through the material at their own rates and with a large amount of independent study. This trial will result in further revision, particularly of exercise and illustrative material, resulting in a book suitable for field testing and experimentation. A senior author and the program director are responsible for consistency of language and notation used in the series, for articulation between the books, and for decisions regarding extent and types of revisions. A general editor supervises the book production. His responsibilities include preparing a detailed index for each book, cross-indexing between books, uniformity of format, consistency of style, relevant introductions, notes, etc.

A total of about 15 books will comprise the EM series. A detailed description of Books 0 through 12 is given in Appendix V. The titles of Books 0-14 are given below:

Book 0 - Intuitive Background
Book 1 - Introductory Logic
Book 2 - Logic and Sets
Book 3 - Introduction to Field Theory
Book 4 - Order in Fields
Book 5 - An Introduction to Mathematization: A Theory of Voting Bodies
Book 6 - Relations
Book 7 - Functions
Book 8 - Number Systems
Book 9 - Finite Probability Spaces
Book 10 - Elements of Geometry
Book 11 - Groups and Rings
Book 12 - Linear Algebra with Trigonometry
Book 13 - Introduction to Real Analysis I
Book 14 - Introduction to Real Analysis II

ROLE OF TEACHER AND GENERAL METHOD OF INSTRUCTION

Students entering the Elements of Mathematics Program at about 7th-grade level begin with Book 0, Intuitive Background, and Book 1, Introductory Logic. They meet in regularly scheduled periods of 60 minutes, five days per week. They work with Book 0 approximately three days per week and Book 1 two days per week. CSMP and the Carbondale school systems have an arrangement whereby the EM students are bussed daily to the CSMP offices for their mathematics instruction.

Initially, instruction is in the typical classroom format. The children are accustomed to this in elementary school. Book 1 is designed, however, for the student to use by himself and, as the child progresses through Book 1, the amount of classroom instruction decreases steadily.

The students have access to study carrels where they may work by themselves or with students at adjacent carrels. Teachers are available for consultation. There are rooms, equipped with blackboards, suitable for small-group work or with an instructor. Students may freely use the library resource center.

The materials stimulate the student to produce written material. Much of the written work is in response to problem sets that appear at regular intervals in the books. Student work, whether produced in school or at home, is turned in each day and is promptly checked and returned to the student. CSMP employs well-qualified older students to grade the papers of students studying at earlier stages of the program. This has been an excellent source of manpower and has provided the graders with an opportunity for a thorough review.

There is no way to predict how much material a student will produce: the highly motivated student will produce a prodigious amount. Neither is there any way to predict the rate of progress; the students begin to pace their way through material at individual rates fairly early, and no effort is made to keep the students together.

The role of the teacher in this type of instruction differs from his role in traditional instruction. An experienced teacher can make provision for individual differences in the traditional classroom, but it is often at the expense of a large proportion of the class. While the teacher works with one subgroup, the rest of the class will be given something to do, but it will almost necessarily be a blanket assignment that does not take account of different rates of growth. By the nature of classroom instruction, the teacher spends no more time with students who need his services than with those who do not. In individualized instruction teacher-student contacts are more selective. Since the impetus for contact often comes from the student at a time of his choosing, the teacher spends a higher proportion of his time with students who have real pedagogical or mathematical concerns, such as seeking help, obtaining recognition of progress, or wanting to share and discuss mathematical experience. On the other hand, the teacher may initiate contact as a result of diagnosing a student's progress through written work and tests, and may do so without having to make provision for the remaining students.

In instruction of this type the teacher is no longer the sole or even the prime expositor, nor does he quite occupy his customary position as arbiter of correctness. Students accustomed to independent learning, and to independent work on problem sets with no answer keys, can find an unexpected number of correct ways to solve and answer problems, and they learn to develop assurance in the correctness of their work.

It is obvious that the teacher cannot and does not need to prepare lesson plans as he might for traditional classroom instruction. He cannot completely prepare himself on any day to deal with a particular needed portion of the curriculum. He will, in any given period, deal with a far wider portion of mathematics than he would in traditional teaching. This requires that he have a better-than-average command of the subject, and know thoroughly the basic material his students are working with.

The teacher needs flexibility to deal constructively with students at different stages in the curriculum. He cannot expect that at any stage most students will have certain skills and understandings, but the teacher must develop his own techniques for discussing each student's progress by questioning him and by analyzing his responses and questions. The teacher will spend more time finding out how the student works and less time looking at the answers he produces.

RELATION OF EM PROGRAM TO ACTIVITY PACKAGE PROGRAM

If it were the prime objective of the total program to produce researchers in mathematics, then one of the most important groups in writing materials for children and instructing them would be the creative mathematicians. But it is mathematics educators, the secondary and elementary school teachers, who do these things. How do the two groups communicate? Sometimes the words they use in common have quite different semantic meanings or contexts, and each uses some words unfamiliar to the other. The school teacher must be able to do mathematics in some sense, but he must also know how to talk about it. Mathematicians prove things; teachers must cope with inducing students to prove things. This entails some analysis of what a proof is. Mathematicians generally do not need this analysis; teachers generally do. Such analyses are carried out by mathematical logicians. Mathematicians are capable of imaginative leaps into the unknown, inspired guesses, fruitful conjectures. The mathematician uses mathematical intuition, but does not generally need to analyze what it may be, except perhaps for himself. For most teachers the matter of mathematical intuition needs some analysis. Mathematicians who are also great teachers, like Polya, provide such analysis.

CSMP is not, of course, devoted to producing researchers in mathematics, but deals with mathematics education for all students whose ultimate career choices will cover the full range of human occupations. It is not possible to provide explicitly for the future vocational needs of all the students, or, indeed, of very many of them. Curriculum planners are forced to make choices based on considerations of child development, social needs, content of mathematics, etc. With respect to content, one is forced to look for what is fundamental, for what the unifying ideas are. When basic ideas are identified, there still remain the problems of sequence of presentation and of method. The CSMP staff at all levels of the program must be able to talk about mathematics to one another. One wants to avoid, for example, presenting a concept of variable for 9-year-olds that will have to be drastically revised for 15-year-olds.

The EM program is very much concerned with problems of exposition of mathematics, with accurate language for dealing carefully and explicitly with notions of set, variable, open sentence, function, relation, proof, etc. It is written for above average, verbally competent students of ages 12 to 18. The language is as precise as we can make it, and as compact as the linguistic competencies of the students will allow. The program provides a language for communicating within the project about mathematical content. This helps to bring about a common understanding of what concepts are basic and what they are; it helps also to prevent inconsistent presentations of the same concept at different age levels.

We do not know how to design education for all 6- to 12-year-olds that will provide for their specific capabilities and behaviors at the age of 20: the goal is too far away, the branching in possible development too extensive. One needs intermediate goals. The content analysis, the program organization, the curriculum choices, and the EM program methods set requirements for desired capabilities and behavior that provide intermediate goals for the program. We do not mean to imply that the 6-12 program is designed primarily to feed the EM 12-18 program. Indeed, the 6-12 program is just the initial part of a 6-18 program for all students. We do not regard the EM 12-18 program as totally different from the eventual 12-18 program for all students.

The EM program may start a little higher up in the mathematical sequence, provide a richer diet, and proceed a good deal further, but the two sets have a substantial intersection of content. Pacing in the two programs will be quite different, textual materials will differ, manners of presentation will vary, but again with a substantial intersection.

Choosing 12-18 as the age scope for the EM program is to some extent influenced by current school organization and a need for field testing in public schools. Clearly, the 6-12, 12-18 division is arbitrary, and decisions made at any place in a unified curriculum will have repercussions throughout. The design is deemed a practical way to get on with the job.

TEACHER TRAINING FOR EM

Field testing of EM material will require teachers experienced in dealing with students, adequately trained in mathematics, and able to assume the teaching role suitable for the materials and the experiment.

A teacher taking part in this experiment will need a more substantial background in mathematics than that of most secondary school teachers. Serious work in mathematical logic is not normally a part of the high school teacher's background, nor is he likely to have taken substantial graduate courses in mathematics. To assume the role designed for this experiment, the prospective teacher will be required to make substantial adjustments. It would vitiate the experiment should the teacher attempt to teach EM material in the customary way much beyond the early part of Book 1. We have observed one disastrous unauthorized attempt to do so. EM students learn to work with a symbolic language that the teacher will also have to learn. There is some indication that students at ages 12-13 learn it more readily than adults.

It is probably necessary that prospective EM teachers have the opportunity to observe students working with the materials. Teachers accustomed to the usual classroom environment invariably underestimate the ability of good students to progress with materials of this type in an ungraded, independent-study environment.

We propose a training program for experienced teachers and promising teachers-in-training that will include:

a) a substantial training in graduate mathematics;
b) explicit training in EM materials, and study of other modern curriculum developments;
c) first-hand experience teaching and observing students working with EM materials;
d) training in problems of testing, observing, and evaluating such an experimental program, both in the laboratory and in the field.

Such a program was initiated for a small number of teachers in the summer of 1969. As originally conceived, the program was financed almost entirely by NSF, with CEMREL supporting the participants who were not eligible for NSF grants. A three-way cooperative arrangement was worked out, for courses and instructors, involving Syracuse University, Southern Illinois University, and CEMREL. The program was planned for about thirty semester hours, distributed over four summers. (Appendix VIII.A gives a list of the courses suggested for this program.)

The first summer's experience made it evident that the plan, as conceived, was not realistic. The "several summers" commitment that was asked of the participating teachers, and of their school systems, presented too difficult a decision for most. The prospect of cutbacks in NSF teacher training support added a further element of uncertainty to the entire venture.

A more modest plan -- an EM Workshop -- was adopted for the summer of 1970: its details are spelled out in Appendix VIII.B. Essentially, the workshop will be conducted by CEMREL-CSMP without fees or tuition. Stipends for approximately 20 participating teachers will come from the respective school systems or from other sources, such as gifted student programs, Title III funds, etc.

For teachers who plan to use the EM materials at the junior high school level only, one summer workshop will probably suffice. Additional workshops will be organized for teachers from schools who plan to adopt the entire EM curriculum; but all such plans will depend largely on the summer 1970 experience.

Clearly we shall look for participants from schools that seriously anticipate introducing EM materials and that agree to meet a set of minimum conditions dictated by the experience of those who have taught these materials during the last two or three years. These conditions refer especially to the selection of EM students (upper 20%, approximately), to the teaching load imposed on EM teachers (a reduced load or clerical help is strongly recommended), to the size of the EM class (the independent type of work expected of the students suggests a smaller pupil-teacher ratio), and to the request of feedback information to CSMP for purposes of evaluation.

Future teacher training plans are being discussed, together with the relative merits of various models. The availability of adequate funding and the commercial production of the EM materials will largely control both the starting schedule and the geographical spread of the teacher training centers that will need to be developed around the country for the diffusion stage of the EM program.

EVALUATION OF THE EM PROGRAM

CSMP faces the usual problem of using comparative evaluation as a summative device. As in the case of other new curricula, especially in the field of mathematics, the problem of a "fair" test for both the CSMP and a control group is a difficult one. In view of the fact that the EM program differs more widely in content from traditional mathematics than other new mathematics projects, the problem is magnified. Since precision of language is considered very important by the writers of the EM program, and since CSMP students may be more sophisticated in this respect, the terminology and symbolism to be used in test items is a problem additional to that of selecting some common content.

The use of well-normed tests as a basis for comparative evaluation is still possible to some extent, though it is recognized that total scores on these tests will represent measures of ability that are too gross -- for the reasons mentioned above, at least. It is hoped that certain content and language can be specified in advance by an independent analysis and that, by working closely with the Educational Testing Service (whose College Board Tests may be taken as a widely normed, well-researched standardized test), items which fall within this defined class can be used as a basis of comparison. This is still a rather unsatisfactory type of comparison for at least two reasons. The content selected will be only a subset of that covered in traditional mathematics and an even smaller subset of the content within CSMP. Similarly, since CSMP students work at their own speed, there will be a very wide variation within this group in terms of content covered. Of course, similar criticism is leveled at large scale standardized tests in general for these reasons; but the reasons that prompt this criticism are especially pertinent here, and do make for a less than satisfactory comparative evaluation.

Another complication in this or in any other comparative study is the fact that students entering the EM program are atypical of the general student population aged 12-18. In fact, they were selected specifically for their mathematical promise, motivation, and high reading level, since the EM program is aimed -- in a rough way, but approximately -- at the top 20% of students. For this reason, the acquisition of base-line data is necessary so that the variation due to ability at the time of entrance into the program may be controlled.

A second source of comparative data which shows promise is the use of tests used in the International Study of Achievement in Mathematics.[1] In this study, a set of ten tests covering fifteen areas of mathematics was used to compare 132,000 students from twelve countries. Five attitude and two descriptive scales were constructed and a wealth of data is available concerning the relationships between mathematics achievement, type of mathematics (new vs traditional), attitudes, descriptive and biographical data from students and teachers, and a variety of social and administrative factors.

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[1] Husen, Thorsten, (Ed.) International Study of Achievement in Mathematics, Vol. I. New York: John Wiley and Sons, 1967.

To quote just one of many findings of the report: "At each instructional level, students with high achievement in mathematics tend to view it as a 'closed system'."[1] This finding supports one of the causes of dissatisfaction expressed at the beginning of this program plan, and the study provides one criterion against which to measure success in overcoming this dissatisfaction.

It is expected that very detailed longitudinal studies will be carried out on graduates of the program. Though the number of graduates is small, this is a necessary investigation. The kind of success these students attain in university work and, particularly, in future mathematics instruction, the choice of courses and careers they make, the attitudes and interests they manifest -- all relate to some of the goals of the program whose achievement cannot be measured while the student is studying within CSMP. Of course, the interaction of so many different variables which produce the above behaviors can never be completely sorted out. Certainly, until the "experimental" or "enthusiasm" effects and the effects due to high quality of teaching by CSMP staff are removed, this kind of evaluation can be regarded as quite tentative.

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[1] Ibid, Vol. II, p. 154, Cognitive Domain.

Because of the high ability level required of the student, the preparation required by the teacher, and the voluntary nature of the commitment by the individual teacher, large scale field trials in the usual sense of the term are not anticipated in the near future. However, as individual teachers and school systems agree to a trial of the materials, one of the conditions of adoption of the materials will be a commitment to certain tasks involving the collection of data as specified by the evaluation staff and the director. This information may be slow to emerge initially, considering the length of the program, but it is hoped that this kind of continuous evaluation will gradually accumulate a variety of evaluative information from a variety of sources.

CEMREL SUPPORT SERVICES

A number of central support services are provided by the Laboratory to CSMP. The most prominent among these -- evaluation and the educational materials center -- have been adequately described under the appropriate components. Other support functions, such as graphics and reproduction, have been constantly used by CSMP both for the package program and for the Elements of Mathematics series. These other services are also provided by CEMREL -- business management, systems development, public information, and diffusion.[1]

CSMP BY-PRODUCTS

Among the more significant by-products of the program, the following should be singled out:

1) Mathematicians' outlines on various mathematical topics: The Integers, Motion Geometry, Graph Theory, Topology, Space Geometry, Measurement.
2) Activity sequence outlines on Positional Notation, Addition Algorithm, Subtraction, Motion Geometry, Rational Numbers, Probability, Multiplication.

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[1] These services are described in detail in the "Support Services Basic Program Plan" prepared by CEMREL.

3) Position paper on "The Use of Algorithms" and a paper on "Estimation".

4) "Mathematics in the Real World", a series of five 16 mm. color films produced by COMMUNICO in cooperation with CEMREL-CSMP. These films -- each about 40 minutes in length -- may be used to enrich high school and college mathematics classes. They feature Professor Joseph M. Gani, Director of the Sheffield-Manchester School of Probability and Statistics. The five films are: Capital and Interest (mathematics of finance, including capital and interest, with a discussion of geometric series and the exponential function); Curves and Trajectories (velocity and acceleration with the path of projectiles used as an illustration); Randomness and Regularity (the axioms of probability as applied to today's world, including the binomial and Poisson distributions and applications); Decisions and Linear Programming (linear programming with solutions of equations involving inequalities, and practical examples); Numerical Systems and Computing (the binary system with simple examples of numerical analysis and the uses of computers.)

5) "The Teaching of Probability and Statistics", a report of the first CSMP International Conference

held at Carbondale in March 1969. The book (containing the text of the lectures listed in Appendix VI) is published by Almqvist & Wiksell, Stockholm, Sweden.

6) "The Teaching of Geometry", a report of the second CSMP International Conference (March 1970) is in preparation.

7) English translation of Papy's "L'Enfant et les Graphes" and "Minicomputer".

8) "CSMP: Where Students and Mathematics Meet", a 25-minute 16 mm. color film on the Comprehensive School Mathematics Program.

9) "Operational Systems Games", by Kaufman and Steiner, McGraw-Hill, Inc., Webster Division, 1969.

10) A "K-1 Report" on existing programs (cf. p. 121).

11) A number of articles published in various periodicals, such as Educational Studies in Mathematics, American Education, The Mathematics Teacher, Croft Educational Services, etc.

12) A "Systems and Procedures Manual" for teachers using CSMP materials.

13) A report on "Testing" for the CEEB.

14) Several documents pertaining to a program of Didactics within the SIU Mathematics Department.

The development, pilot testing, and revision processes at CSMP have yielded these and a number of other outcomes for which some independent use may well soon be found either in the form of useful by-products or as a basis for educational studies and research papers.

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SUMMARY
AND PLANS OF CSMP ACTIVITIES

- Administration
- Content Selection and Analysis
- Package Program
- EM Program

SUMMARY AND PLANS OF CSMP ACTIVITIES

The purpose of this section is to review the accomplishments of the past year, to outline the activities of the year ahead, and to lay out CSMP's plans over the next few years.

While the 1969-70 accomplishments and the 1970-71 activities can be set down with a good degree of accuracy, the plans for the future can only be outlined in a very tentative fashion, contingent as they are upon funding realities and upon significant decisions that will have to be made vis-a-vis those realities. These decisions will have particular bearing on how far teacher training plans and the field-trial stage of the program will extend outside the immediate local scope of CSMP's work, since the assured continuance of the present level of funding will provide for the development and pilot testing of CSMP materials within a very limited local setting only.

After an introductory section on general administration, CSMP's accomplishments (1969-70), proposed activities (1970-71), and plans (1971-76) will be presented for each of the three components described in this Basic Program Plan.

*

ADMINISTRATION

The administrative tasks of CSMP are carried out by the director, two assistant directors, an administrative assistant, and a staff member charged with technical writing, general editing and public relations. These constitute CSMP's administrative council and they hold planning and ordinary program operation meetings approximately three times a week.

Administrative discussions and decisions this year have dealt in particular with the following matters: scheduling, planning, conducting, and reporting five Staff Associates meetings (cf. Comp. I below); revising and updating the Basic Program Plan; writing of ordinary and ad hoc reports on the program (such as the Contractor's Request, a 1970 questionnaire for the International Clearinghouse Report, a short overview of the full CSMP operation for an NSF pre-proposal paper, etc.); planning a detailed report of the 1969-70 pilot study; finalizing allocation of staff and overall plans for the new school year's operation to begin this summer; accommodating and hosting a steady flow of scholars and visitors interested in the program; preparing an up-to-date display of CEMREL and CSMP materials; re-analyzing the disposition of the budget, adjusting it to cuts, and developing a monitoring system for budgetary procedures; keeping the educational and mathematical community informed and involved in the program's progress through lectures and presentations at local, regional, and national meetings (cf. CSMP's Quarterly Reports); searching for, and hiring, eight new staff members; working out administrative details with local school systems for class schedules and transportation of 150 EM students and for the extended pilot study to be held in Carbondale; planning summer teachers institute and workshop for EM teachers; exploratory meetings with SEDL[1] personnel with a view to establishing cooperative arrangements between the two programs; working out administrative details with COMMUNICO for the 5-film series on "Mathematics in the Real World"; planning and implementing the details involved with transportation and accommodation of participants and visitors at the CSMP International Conference on Geometry.

One of the major tasks undertaken and completed this year was a complete reorganization of CSMP's administrative structure and of the program's organization for package production. The details for the latter are given in the description of the appropriate component (cf. Component II).

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[1] Southwest Educational Development Laboratory, Austin, Texas.

COMPONENT I: CONTENT SELECTION AND ANALYSIS

1969-70 Accomplishments

Without a doubt, one of the most significant accomplishments of CSMP this year is the continued interest and involvement of dedicated mathematicians in the program's general orientation and in its immediate concerns.

It is through actual contact and discussions with the program's staff that the mathematicians' involvement continued to be realistic and eminently practical.

The members of the National Advisory Committee gathered for a week end in Carbondale on the occasion of the December 1969 Staff Associates' meeting. Most of them returned again for the International Geometry Conference.

The program's mathematical strength is provided by CSMP's group of Staff Associates. The SA structure was radically reorganized during this year. Five meetings were held (October and December, 1969; January, April and June, 1970). Specific assignments were agreed upon for the preparation of mathematicians' outlines for package development. First drafts of the outlines on motion geometry, space geometry, and measurement were prepared, as well as revised drafts of the outlines on the integers, on topology, and on graph theory. A position paper on the use of algorithms was also completed. The co-chairmen for the 1971 International Conference on Algebra were selected (Peter Braunfeld and W. E. Deskins) and preliminary planning work was begun. A staff associate (Vincent Haag) was hired as full time Staff Associate Activities Coordinator (beginning September 1970). At the April meeting of the Staff Associates serious work was begun toward a K-6 content development master plan. A small Staff Associates documentation center was set up in the CSMP mathematicians' suite.

CSMP was successful in obtaining more extended periods of service in Carbondale from several Staff Associates and other mathematicians (Braunfeld, 14 weeks; Deskins, 6 weeks; Haag, 12 weeks; Krause, 2 weeks; Råde, 7 weeks; Shanks, 2 weeks; Sterling, 2 weeks; Steiner, 12 weeks; Troyer, 1 week. Staff Associate Robert Exner devotes four days each month to writing EM material. Arthur Engel, 16 weeks; Madame Papy, 3 weeks; Claude Schochet, 3 weeks; Ingo Weidig, 12 weeks.) Seminars on various topics (e.g., Graph Theory, the Simulog computer, etc.) were held.

Discussions have been held regarding a proposal to explore the possibility of expanding the Mathematics Department's offerings to include a program of Didactics at Southern Illinois University. The SIU Mathematics Department voted favorably on the proposal.

A report was issued on a study of exemplary development programs (Minnemath, Nuffield, Papy, Dienes, etc.) with a view to fit into the existing CSMP package structure some appropriate existing K-2 materials.

A mathematics librarian was hired: a new functional classification system for the program's mathematical holdings was developed and work was continued towards developing a more complete documentation center.

Of great significance to the CSMP's content component was the Second International Conference, on the teaching of geometry, which attracted 25 participants and 150 observers. The Conference proceedings will probably be published by Almqvist & Wiksell, Stockholm; editing, typing, and proof-reading of the original manuscripts have already begun.

Of relevance to this component are also the items listed under CSMP's by-products (pp. 109-111): the translation of Papy's "L'Enfant et les Graphes" and "Minicomputer" into English; the publication of the 1969 International Probability Conference Proceedings: the completion of a 5-film series featuring Professor Joseph M. Gani, Director of the Manchester-Sheffield School of Probability and Statistics. The series was filmed and produced by COMMUNICO in cooperation with CEMREL-CSMP.

1970-71 Proposed Activities

Regular meetings of the CSMP staff associates will continue to be held (October, December, 1970; January, April, and June, 1971). Priority will be given to the completion of the master plan for K-6 content development outlined at the April 1970 meeting.

Writing plans call for a first draft of a mathematician's outline on the Rationals and on Number Theory, with continuing revision of the existing outlines. Also, teachers' guidelines for the K-2 program will be prepared.

With the hiring of a full-time staff associate activities coordinator, regular staff seminars can be scheduled more easily, while the resident mathematician structure can be studied and solidified.

Discussions and negotiations will continue toward possible collaboration with Southern Illinois University on the establishment of a program of Didactics within the Mathematics Department. (Cf. p. 120.)

The proceedings of the International Conference on Geometry will be edited and published. Work on the Third CSMP International Conference on the Teaching of Algebra, to be held in March 1971, will continue. Preliminary plans will be made for the 1972 conference on Logic.

Depending on available funding, CSMP plans to continue the mathematics film series begun this year (cf. p. 110).

Future Plans: (1971-76)

The content component will increase in scope and importance when CSMP will begin to concentrate at the secondary level. Specific details will be worked out in the 1974-75 proposals, but long-range planning and discussion will continue to occupy both CSMP staff and staff associates. The secondary level program will also require appropriate mathematicians' outlines for package development and a carefully planned fusion of the EM series with the activity packages.

The wider scope and greater importance of the content component at the secondary level will entail greater efforts on the part of CSMP to keep mathematicians involved in the development of the program.

The series of annual international conferences will continue. A second round of such conferences will need to be planned for the purpose of reviewing the basic topics already covered to update information and expand the study to meet CSMP's new needs and concerns.

To help keep CSMP's curriculum realistically attuned to society's needs, plans are underway to hire Professor Arthur Engel some three months per year to review the entire program and embed appropriate and relevant applications into the already developed mathematical content.

Both behavioral and physical aspects of computer-oriented programs will be studied to fit them into the CSMP curriculum over the next five years.

The various alternative approaches to content described under Component III (p. 72) will be studied and evaluated.

The film series begun in 1969-70 will be continued and expanded, depending on available funding.

COMPONENT II: ACTIVITY PACKAGE PROGRAM

1969-70 Accomplishments

The changes in CSMP's package production organization (cf. pp. 49-55) together with the adoption of a simplified two-year "activity development-pilot testing" plan (cf. CSMP's scope of development chart at the end of this section) should be singled out as among the most significant accomplishments in the implementation of Component II this year. One immediate outcome was the formulation of an eight-year plan for the development, production, and diffusion of CSMP's elementary-level materials.

Approximately 170 activities (cf. pp. 55-56) were completed for children of ages 8-9. CEMREL support was provided through the Laboratory's graphics and reproduction services. Team A (cf. p. 55, note) had as its main responsibility the development of the activities: they attended regular seminars on various mathematical topics (topology, graph theory, motion geometry, etc.) conducted by CSMP's staff associates; they taught a small development group of children (Class A) while developing 3rd-grade materials, and at the same time planned 4th-grade activities. They prepared activity sequence outlines on such topics as addition algorithm, positional notation, subtraction, motion geometry, probability, multiplication, rational numbers, etc.

A pilot study of 3rd-grade materials was organized in four Carbondale classrooms, while CEMREL continued to provide evaluation support in the gathering and interpreting of collected data. A complete report of the 1969-70 pilot study is in preparation and will be ready in October. Plans have also been finalized to extend the pilot study of CSMP 3rd-grade materials, and concomitant management system, to a dozen classes in the fall of 1970.

As part of the pilot test effort, the position of a full-time pilot study coordinator was established, a revision team (for 3rd-grade materials) was formed and has been operating since early February, in-service seminars and a summer workshop were organized for pilot study teachers, while suitable models for pilot teacher training were studied and discussed.

A paper was prepared with recommendations for the K-1 development program which is to begin in September, and a K-1 team was formed to implement these recommendations. A report was also issued on exemplary development programs in an effort to see if some existing materials could be fitted into the CSMP primary curriculum structure. Some staff members visited the LRD Center at the University of

Pittsburgh to examine the PEP program for possible use in the CSMP K-1 curriculum.

A six-week pilot study was held with 2nd-graders at University School to test revised materials and to modify the classroom management system.

Lastly, a number of seminars with Madame Frederique Papy were planned (to be held in June and July) for the CSMP writing staff, with possibly some demonstration classes with primary students. Mme. Papy is known for the experimental work with 6-year olds which she began in 1967 at the Belgian Center on the Pedagogy of Mathematics.

1970-71 Proposed Activities

Team A will continue to develop a preliminary version of 4th-grade level activities while teaching a group of 4th-grade students.

The activity sequence outlines will undergo constant revision.

Increased output is expected from the recently formed revision team. Pilot testing of 3rd-grade materials will extend to 12 Carbondale classrooms for the purpose of trying out the revised activities as well as studying various models of a classroom management system. Plans call for a continuation of the pilot study of 3rd-grade materials in 1970-71: this will make it possible for us to pilot test 4th-grade activities in 1972 on a group of 4th graders already exposed to the CSMP program. The financing of such a plan is still under discussion.

With a full-time staff associate in residence it will be possible to hold regularly scheduled seminars for the CSMP staff.

Specifications for teacher training for the field trial and diffusion stages of the program will be written.

The K-1 materials will be readied for public school use in September 1971.

Negotiations between CEMREL and SEDL will continue toward possible cooperative arrangements between their two mathematics programs.

Procedures for more efficient use of the CEMREL support services will be worked out, especially with the graphics and production departments.

Future Plans (1971-76)

The 2-year "activity development - pilot testing" plan outlined in the CSMP scope of development chart at the end of this section will continue to be implemented.

In accordance with this plan, 4th, 5th, and 6th grade activities will be ready for pilot testing in the fall of 1972, 1974 and 1976, respectively. The number of schools involved will be the same as for the 1970-71 program described above, with extended pilot tests contingent upon increased funding.

Second-grade level materials will be readied in the 1971-72 school year, so that it will technically be possible to guarantee a complete CSMP K-6 package program to children who begin the first grade in 1972.

The need for statistical data and summative evaluation can only be met through carefully planned field trials. Funding restrictions have forced us to abandon, for the time being, actual negotiations for such trials. However, we shall make every effort to use our small-scale pilot-test and teacher-training experience to develop appropriate plans should the funding situation radically change.

Beginning of the fusion of the EM and Package programs is planned for 1977.

COMPONENT III: ELEMENTS OF MATHEMATICS PROGRAM

1969-70 Accomplishments

The first draft of chapters 5, 6, 8, 9, and 10 of Book 0 was completed, and class trials continued for chapters 1-6 and 9-10.

The revision of Books 1 and 2 is finished; each book has been provided with an answer book and a set of quizzes. A set of quizzes has also been prepared for Books 3 and 4 which are now being revised.

Book 5 has also been revised; an answer key and a set of quizzes have been written for Book 6.

The manuscript for Book 7 (except for one section) is ready; substantial parts of Book 8 are in rough manuscript form; with the completion of the planned "Supplement", Book 9 is now finished. The outlines for Books 10 and 12 are ready and the writing of Book 10 has begun. The first rough manuscript of Book 11 is finished.

Classroom trials of the pilot version of the ten EM books has continued for about 150 Carbondale Junior and Senior High School students.

Staff discussions were held on EM classroom management and feedback procedures.

A summer institute for EM teachers was held at Syracuse University.

1970-71 Proposed Activities

Class trials will continue (for approximately 170 students) of Books 0 through 11.

Indexing and general editing of each book of the series will continue.

First drafts will be prepared of Chapter 7 (Book 0), and of Books 7, 8, 10, 11, and 12.

Revision of Books 0, 3, 4, and 6 will continue, and outlines for Books 13 and 14 will be prepared.

Additional classes, other than in Carbondale, will offer the EM program: at this writing, between 6 and 12 classes are planning to use some of the CSMP texts.

A summer workshop for EM teachers will be conducted in Carbondale, and plans will be finalized for the following summer's workshop.

Future Plans (1971-76)

The planned EM series will be completed, including editing, revision, and classroom trials for each of the books, with the possible addition of some supplementary topics.

A greater number of schools, outside the local Carbondale setting, will be included in the extended trials of the EM series.

CSMP will develop specifications for the training of EM teachers, will plan institutes for them, hopefully involving the cooperation of colleges and universities. Teachers manuals to accompany the EM series will be prepared, depending on need and available funding.

The possibility of developing video tapes and other media to supplement EM teachers and texts will be carefully studied.

Beginning of the fusion of the EM and Package programs is planned for 1977.

APPENDIX I

* CEMREL Administrative Staff

* 1970-71 CSMP Staff

* CSMP Staff Associates

* CSMP National Advisory Committee

* CEMREL Advisory Committee on Evaluation

* CEMREL Board of Visitors

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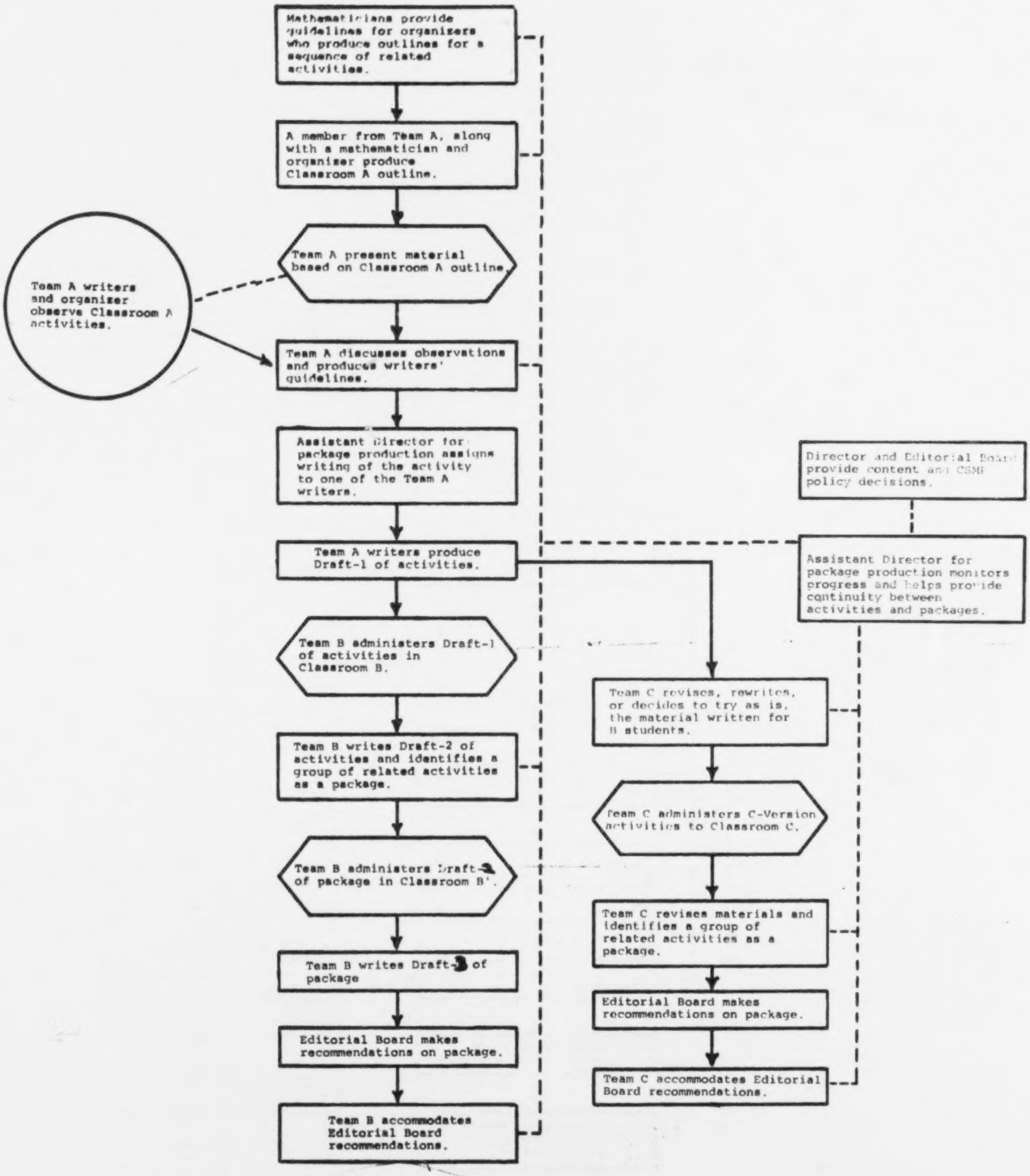
APPENDIX II

A: 1968-69 Organization Chart for Activity Package Production.

B: 1969-70 Organization Chart for Activity Package Production with detailed description.

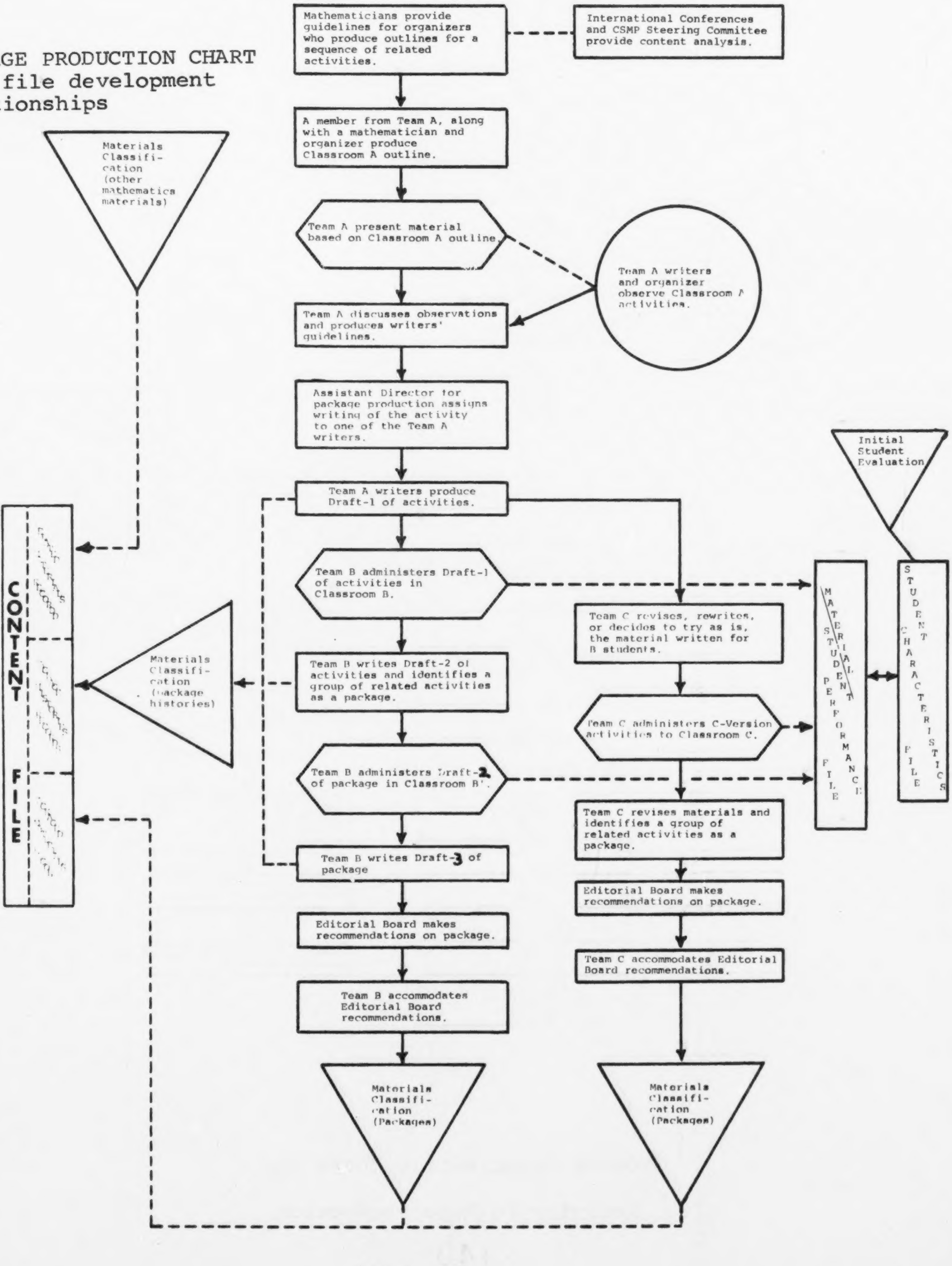
APPENDIX II.A


Mathematicians provide guidelines for organizers who produce outlines for a sequence of related activities.
A member from Team A, along with a mathematician and organizer produce Classroom A outline.
Team A present material based on Classroom A outline.
Team A writers and organizer observe Classroom A activities.
Team A discusses observations and produces writers' guidelines.
Assistant Director for package production assigns writing of the activity to one of the Team A writers.
Team A writers produce Draft-1 of activities.
Team B administers Draft-1 of activities in Classroom B.
Team B writes Draft-2 of activities and identifies a group of related activities as a package.
Team B administers Draft-2 of package in Classroom B'.
Team B writes Draft-3 of package
Editorial Board makes recommendations on package.
Team B accommodates Editorial Board recommendations.
Director and Editorial Board provide content and CSMP policy decisions.
Assistant Director for package production monitors progress and helps provide continuity between activities and packages.
Team C revises, rewrites, or decides to try as is, the material written for B students.
Team C administers C-Version activities to Classroom C.
Team C revises materials and identifies a group of related activities as a package.
Editorial Board makes recommendations on package.
Team C accommodates Editorial Board recommendations.

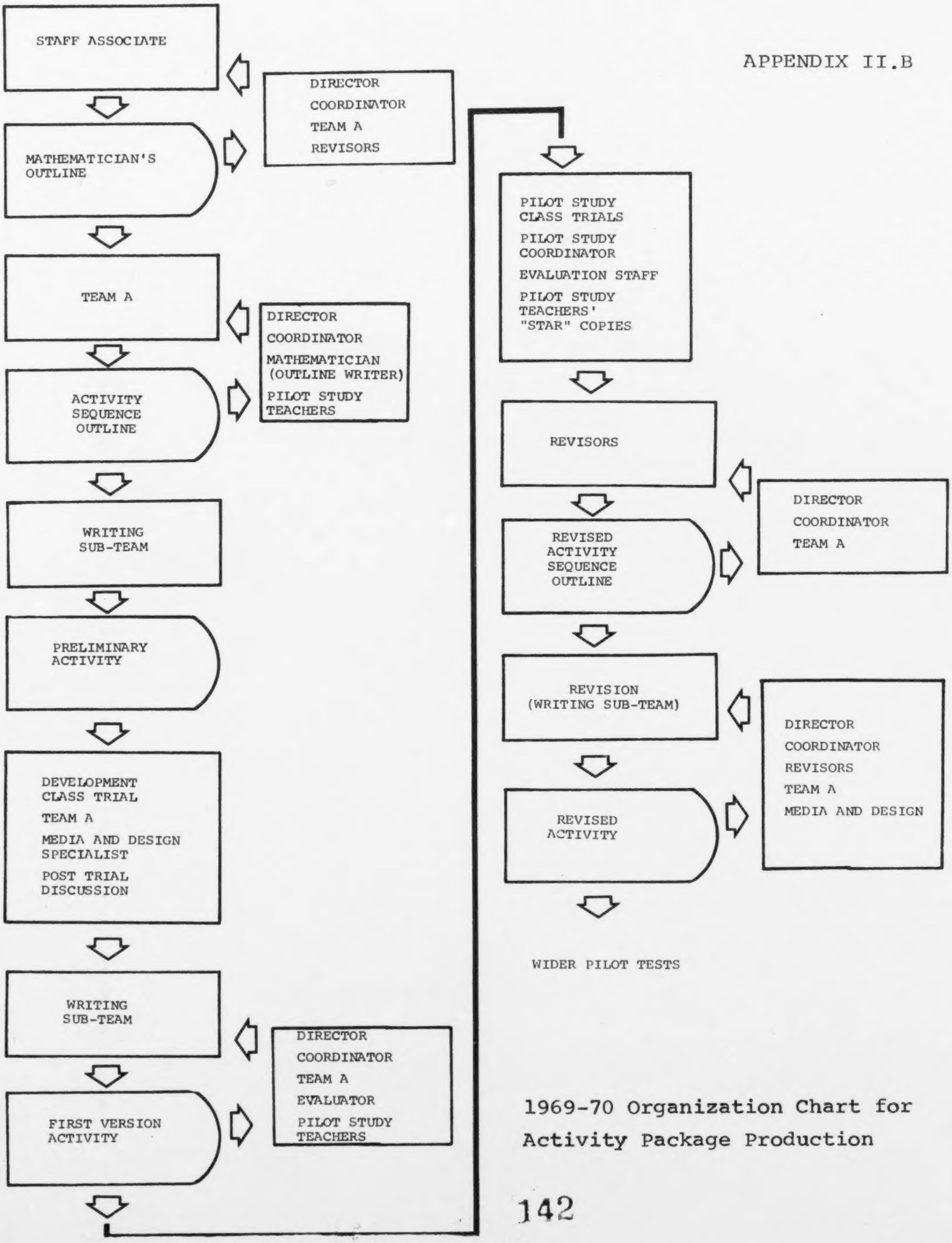
1968-69 Organization Chart for

Activity Package Production

PACKAGE PRODUCTION CHART with file development relationships


Mathematicians provide guidelines for organizers who produce outlines for a sequence of related activities.
International Conferences and CSMP Steering Committee provide content analysis.
A member from Team A, along with a mathematician and organizer produce Classroom A outline.
Team A present material based on Classroom A outline.
Team A writers and organizer observe Classroom A activities.
Team A discusses observations and produces writers' guidelines.
Assistant Director for package production assigns writing of the activity to one of the Team A writers.
Team A writers produce Draft-1 of activities.
Team B administers Draft-1 of activities in Classroom B.
Team B writes Draft-2 of activities and identifies a group of related activities as a package.
Team B administers Draft-2 of package in Classroom B'.
Team B writes Draft-3 of package
Editorial Board makes recommendations on package.
Team B accommodates Editorial Board recommendations.
Materials Classification (Packages)
Team C revises, rewrites, or decides to try as is, the material written for B students.
Team C administers C-Version activities to Classroom C.
Team C revises materials and identifies a group of related activities as a package.
Editorial Board makes recommendations on package.
Team C accommodates Editorial Board recommendations.
Materials Classification (Packages)
Materials Classification (other mathematics materials)
Materials Classification (package histories)
CONTENT FILE
Initial Student Evaluation
MATERIAL STUDENT PERFORMANCE FILE
STUDENT CHARACTERISTICS FILE

APPENDIX II.B


STAFF ASSOCIATE
DIRECTOR
COORDINATOR
TEAM A
REVISORS
MATHEMATICIAN'S OUTLINE
TEAM A
DIRECTOR
COORDINATOR
MATHEMATICIAN (OUTLINE WRITER)
PILOT STUDY TEACHERS
ACTIVITY SEQUENCE OUTLINE
WRITING SUB-TEAM
PRELIMINARY ACTIVITY
DEVELOPMENT CLASS TRIAL
TEAM A
MEDIA AND DESIGN SPECIALIST
POST TRIAL DISCUSSION
WRITING SUB-TEAM
DIRECTOR
COORDINATOR
TEAM A
EVALUATOR
PILOT STUDY TEACHERS
FIRST VERSION ACTIVITY
PILOT STUDY CLASS TRIALS
PILOT STUDY COORDINATOR
EVALUATION STAFF
PILOT STUDY TEACHERS' "STAR" COPIES
REVISORS
DIRECTOR
COORDINATOR
TEAM A
REVISED ACTIVITY SEQUENCE OUTLINE
REVISION (WRITING SUB-TEAM)
DIRECTOR
COORDINATOR
REVISORS
TEAM A
MEDIA AND DESIGN
REVISED ACTIVITY
WIDER PILOT TESTS

1969-70 Organization Chart for Activity Package Production

ACTIVITY PACKAGE PRODUCTION (Cf. 1969-70 Chart, Appendix II.B)

A mathematician's outline on a specific topic (e.g., Graph Theory, Topology, The Integers) begins to take shape at various conferences with the staff associates, the program director, and the members of Team A.[1] It is during these general conferences that priorities are established, a broad content description agreed upon, and suggestions of possible activities discussed; the rationale for inclusion of such content at a particular level (3rd grade) is also outlined. After these meetings an individual staff associate is charged with actually writing a first draft of the outline. This document is prepared in accord with a set of practical "Guidelines" drawn up to ensure that the outline will indeed provide the needed mathematical guidance for the writing staff during the development of the materials (see Appendix III.A). The staff associate's work is only beginning, since he will play an important role throughout the package development in an advisory capacity and as content editor for activities developed from the outline he is writing. He also suggests overall goals, and approves such goals as may be part of the various package activities.

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[1] "Team A" was originally chosen as the name for the group of teacher-writers working with Class A students (cf. p. 51). The name is being kept even though Class B and C have been discontinued.

The first draft of the outline is then sent for review to other staff associates, as well as to the program director and Team A members for discussion. The coordinator of staff associates' activities plays the rather important role of organizing all the comments made about the outlines by other staff associates and by Team A. He also helps in coordinating various mathematicians' outlines, and informs both staff associates and the CSMP staff of approaches that may have been overlooked, of pertinent literature, and of other experimentation along the field of concern.

All comments from these various groups ultimately flow back to the staff associate who authored the original outline (as well as to the SA file for later reference and discussion). The author then works these comments into the second draft of his outline. (See Appendix III.B for a sample mathematician's outline on The Integers.)

After the second draft of the outline is written, it goes back to the other staff associates, not necessarily for further comment, but mainly for information.

Once the mathematician's outline is available, the Team A members meet with the director as often as necessary (usually twice a week) in order to find ways of breaking down the mathematical content of the outline into pedagogically manageable units which will eventually develop into an activity package. Questions about long-range and immediate goals, about articulation with existing packages, about suitability of activities and their sequencing, are usually brought up and discussed at these meetings. The result is a first draft of what is called an "activity sequence outline", together with pertinent goals. (For a sample activity sequence outline on Motion Geometry see Appendix III.C). This draft is carefully read and commented upon by a number of concerned people within the organization: the coordinator, the pilot study teachers,[1] the teacher-writers, the directors, and the staff associate responsible for the original outline. All comments and suggestions are then put into the hands of some members of Team A, chosen on the basis of their particular interest, to do the writing of the package itself. As this subteam begins to concentrate on the physical planning of the printed activity, a revised sequence outline is produced, together with an activity for Class A (the development group of students that Team A meets with and teaches daily). This constitutes the very first contact of student with content. At this point the activity is in the rough: it might be at first a teacher-taught lesson, with children working and interacting at their desks or at the blackboard, so that a reasonable starting point can be established for writing the activity.

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[1] Cf. p. 63.

If it is a written activity, the students are usually observed by a teacher-writer and a member of the media and design department. Each observes one or two students and, as they are working, jots down pertinent comments on a copy of the same activity. Immediately after the class is over, the teacher-writers and the media and design representative meet for discussion. As a result several pages of a particular activity may develop into a second separate activity. This is usually the work of the subteam responsible for the original activity. It is they who selectively incorporate pertinent comments into what is called the "pilot study activity, draft one".

Another area of concern at these meetings is the choice of an appropriate sequence. Here Team A decides where the particular activity best fits into the total package picture in terms of prerequisites, level of difficulty, etc. One package may consist of 20 activities, another package of maybe a dozen. A student will not take all the activities of a package in, say, multiplication, and then go on to a package in subtraction; he will be exposed to a mixture of activities from a number of different packages. Another consequence of Team A discussion is a revised activity sequence outline, in which the goals of each activity can be better described, together with a sketchy outline of each activity within the package. A rough draft of the pilot study activity then goes to the evaluator, the pilot study teachers, the other Team A members, the directors, and the staff associates. The evaluator prepares a first draft of pre-tests and post-tests for the package being developed. All comments are then collected and are looked into very carefully by the subteam who is primarily responsible for the activity. It is really their task to decide which comments should be incorporated and which should not. The activity is now ready for the media and design department. Due to the fact that activities from several packages may be at this same stage of development, an assistant director establishes the production priorities. The media and design department now produces the "pencil copy".

This last rough copy goes from the media and design department to the subteam (of Team A) responsible for it. Copies are made and routed to the directors for a content check, and to the pilot study coordinator for a format check. As a matter of fact, few problems are encountered at this late stage in the content check. But there might be some difficulties in the matter of format: cluttered title pages, sequencing directions that were not too clear, manipulatives demanded by the activity not easily identifiable on the printed page, answer blanks that were too small, etc. One formal problem had to do with the stylized printing used by the media and design department: it was difficult for third graders still learning to read. First, an elementary typewriter was used, but its spacing was poor; the problem was solved when another typeface was chosen which was fairly easy to read. (Cf. App. III.D for sample activity in Motion Geometry.)

After the content and format checks, the pencil copy returns to the subteam for last changes and accommodations, should they be required. It is now ready to go to the media and design department for the final, or "ink", copy which is given a last check before going to the pilot study coordinator, who immediately sends it to production either in Carbondale or in St. Louis. When copies return, the writers make up a "teacher's edition" and select some pages to be used as remedial material for students who may have difficulties going through the activity the first time. The material is then brought to the pilot study classrooms and placed into the hands of the students. Teachers' comments are collected on what are known as "star copies". (There is a "star copy" for every pilot class.) These star copies go to the revision team along with pertinent general comments.

APPENDIX III

A: Guidelines for Writing Mathematicians' Content Outlines.

B: Sample Mathematician's Outline on "The Integers".

C: Sample Activity Sequence Outline on "Motion Geometry".

D: Sample Activity in Motion Geometry.

APPENDIX III.A

GUIDELINES

FOR WRITING THE MATHEMATICIANS' CONTENT OUTLINES

I. GENERAL CONTENT DESCRIPTION

A general content description is needed which may have implications for a full K-12 program. One must bear in mind that a K-12 program of the future will be more extensive than any presently available.

We hope that you will respond to some of the following:

- Can you give a brief description of the content of your topic? What are the major concepts? What does one actually study?
- You may want to give a few sample problems at different levels of sophistication.
- How is this content related to other areas of mathematics?
- What are some of the most important applications of this content within mathematics? In other areas?
- What are some specific instances which show how this content has been applied within physics, chemistry, biology, the humanities, the social sciences, linguistics, etc.?

II. DETAILED DESCRIPTION OF CONTENT APPROPRIATE FOR BEGINNING ACTIVITIES FOR ELEMENTARY SCHOOL STUDENTS

This will be our guide for writing activities during the coming year.

Reconsider the questions in I. while writing a detailed description of the content for beginning activities.

We will also need responses to the following:

- Where should we place our major emphasis in terms of the experiences we give youngsters in these beginning activities?
- Are there danger areas you would like to comment upon, any warnings, restraints, frequently misunderstood or misinterpreted ideas or concepts, etc.?
- Are there alternate approaches you could suggest and give justifications for?
- Should we use more than one approach with the same students?
- Can you give any particular justification for doing this?
- If there are approaches to this content that you would not like to see in our program, please comment.

III. RATIONALE

Please help us answer some of the following questions:

- Why should this content area be considered for the K-12 curriculum? We are sometimes asked why we spend 2 months on particular concepts in the elementary school when they could be dealt with in 2 weeks at the secondary level. Give reasons that would justify the inclusion of this content in our program.
- Why do you think it is desirable to introduce this content in our elementary school curriculum?

We may find that our original attempts to present some ideas fail. Are there ideas for which we should search extensively for a successful mode of presentation? If so, please comment on their importance. (For example, some of these ideas may be essential prerequisites for future content.)

- Do you feel there is a core curriculum? If so, which concepts in this outline are part of it at the elementary school level?

- Is this content important
  - because of its relation to other content?
  - because it can be blended with some other area of mathematics, thereby making independent presentations unnecessary?
  - because of the methodology and mode of reasoning it develops in the students?
  - for other reasons?

IV. SUGGESTED ACTIVITIES AND PROBLEMS

What are some specific activities and problems for K-6? (Do not exclude those you think would challenge even the most talented students of this age group.)

These activities may be of various types:

- open-ended problems, i.e., problems the student might not solve immediately, but may come back to frequently over some period of time;
- problems which may be strongly motivating because they demonstrate important applications;
- problems for which the student must invent his own tools or methods, as opposed to exercises which are only for practice;
- problems which incorporate concepts from other content areas.

Include references to ideas in existing materials, and be on the lookout for ideas we can use from other curriculum projects.

Specify any language or notation you think should be introduced within these activities. It would be helpful if you would comment regarding consistency and standard usage of language and notation.

V. BIBLIOGRAPHY

What are good sources for our understanding of the content? Include also references to the theory. Indicate one or two references which give a satisfactory overview of the content.

What are good sources for applications, activities and problems?

If there are different approaches to the content, what good materials (both elementary and sophisticated) make use of these approaches?

VI. MATHEMATICAL GOALS

Please spell out what you think are:

- short-range goals,
- medium-range goals,
- long-range goals.

APPENDIX III.B

MATHEMATICIAN'S OUTLINE FOR PACKAGE DEVELOPMENT*

Introduction to the Integers

I. Why introduce integers in the package program?
   A. Role of integers in mathematics. . . . . p. 1
   B. Need for integers in a primary program . . . . p. 1

II. Choice of approach to integers in the package program.
   A. Some possible approaches to Z
      1. Integer as a mapping or operator. . . . . p. 2
      2. Integer as a pseudo vector . . . . p. 3
      3. Integer as an equivalence class of $W \times W$ . . . . p. 3
      4. Integer as an element of $(\{\wedge\} \times N) \cup W$. . . . . p. 4
   B. Comparative analysis of the approaches . . . . p. 6
   C. The recommended approach in the package program. . . . . p. 8

III. Suggested activities in the integer packages
   A. Development of $(Z,+,<)$
      1. Motivating and introducing the integers . . . . p. 10
      2. Addition in Z . . . . p. 12
      3. Applications and concepts . . . . p. 13
   B. Notation . . . . p. 15
   C. Subtraction and multiplication in Z . . . . p. 16
   D. Prerequisites and parallel packages . . . . p. 24

IV. Short and long term goals . . . . p. 25

V. Bibliography . . . . p. 26

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* **This is the original typescript from Prof. Vincent Haag, author of the outline.**

I. Why introduce integers in the package program?

A. Role of integers in mathematics

One of the major goals of any mathematics program must be an understanding of the structure of the real number system R and an ability to apply the properties of operations in R and functions on R to a variety of situations. Whether the real number system is constructed from the set W of whole numbers, or postulated, or both, the subsystem Z of integers should appear as a distinguished subsystem at one step of the construction, or should be characterized by a subset of the defining properties of R, or both.

There are two primary reasons why Z should be singled out for study.

i) The structure of Z provides some of the first and simplest examples of the algebraic properties of various algebraic systems (eg., it is an example of a group, a commutative group, a cyclic group, the infinite cyclic group, the free group on one generator, a ring, a ring with unit, a commutative ring, an integral domain, a Euclidean domain, a Gaussian domain, a Noetherian ring, etc.)

ii) Z provides a system in which not only addition and multiplication are operations, but also subtraction, thereby admitting a process of "counting down" as well as "counting up" and allowing for solutions of $a+x = b$ in the computations related to natural problems.

B. Need for integers in a primary program.

A young child begins to abstract from physical models to numbers when he thinks of the number 3 as a simulator of all situations in which three objects are involved. This process is extended when he accepts a new kind of number as a simulator of situations in which a state might be described as 3-off or 3-below, as opposed to 3 (meaning 3-on or 3-above). By this time he has seen how a part of the line can be coordinatized with W and how elements of W correspond to "on" or "above" states as he "counts up" on the number line. When he considers problems suggesting that he "count down", thus taking him to points on the line not yet labeled, it would seem unnatural not to assign new numbers to these points.

In a primary program we should either do nothing with problems suggesting negative states or else we must be prepared to abstract to negative numbers whenever we deal with such problems. The latter should be possible in the third or fourth grade.

Thus, by the fourth grade, there should be some experience with computations in (Z,+), but it is difficult to make a case for subtraction or multiplication in Z before there is enough sophistication to deal with these operations from a strictly structural point of view. This might not occur until the secondary program for some children, if ever. This matter will be discussed in section IIIc of this outline.

II. Choice of approach to integers in the package program.

A. Some possible approaches to Z

Given the set W, what should be the meaning of Z in the package program? There appear to be a number of alternatives.

1. Integer as a mapping or an operator

There are various ways to define integers as mappings. Among these are:

a. An integer is a translation of a line, in either direction and any distance that is a multiple of a given unit distance. Translations in one of these directions are called positive integers and their inverses are the corresponding negative integers. This approach is equivalent to one in which a positive integer is a translation of the W number line that maps the 0-point onto a point for an element of W. The corresponding negative integer is the inverse of this translation. Since the domains of these translations are the set of points of the line, the inverses exist even though some points of the line have not been given labels. Addition in Z is then defined to be composition of translation, and the integer 0 is the identity translation.

b. An integer is defined by a universal mapping, as in [1] and [5], p. 81. This approach views Z as the unique (up to isomorphism) free group on one generator.

2. Integer as a pseudo vector

In this approach an integer is defined to be an element of the set $\{L,R\} \times W$. An integer is then a pair consisting of one of two "directions" and a whole number "magnitude" (the so-called "signed number" approach). For all $m, n \in W$, define addition in Z by

$$(L,n) + (L,m) = (L, n+m),$$

$$(R,n) + (R,m) = (R,m+n),$$

$(L,n) + (R,n) = (L,o) = (R,o)$, the identity element, with the condition that this addition is associative and commutative.

There are several devices with which one could introduce integers in the sense of pseudo vectors. The Killer game [2] uses armies of red and blue checkers, so that a set of five blue checkers, for example, corresponds to (L,5) and a set of four red checkers to (R,4). Reds and blues kill each other in this game, one-to-one, so that

$$(R,4) + (L,5) = (R,4) + (L,4) + (L,1) = (L,1),$$

and one blue checker survices as the sum of 5 blue and 4 red.

The Postman stories of Davis [3] use bills and checks, so that a bill for $10 corresponds to the pair (L,10) and a check for $15 to the pair (R,15). (L,n) and (R,n) cancel each other, for each $n \in W$, so that

$$(L,10) + (R,15) = (L,10) + (R,10) + (R,5) = (R,5).$$

Davis' "Pebbles in the Bag" stories also are of this type with L standing for "less than" and R for "more than". Putting 4 pebbles in the bag and then taking 7 out corresponds to $(R,4) + (L,7) = (L,3)$; that is, there are now 3 less pebbles in the bag than when we started.

3. Integer as an equivalence class of WxW.

The construction of Z from W follows the usual procedure of imbedding a commutative semigroup with cancellation property (in this case (W,+)) into a group (in this case (Z,+)). The procedure consists of defining a particular equivalence relation $\sim$ on $W \times W$ and then defining Z to be the quotient set $W \times W/\sim$. Addition in Z is then defined in such a way that a subset of Z is isomorphic to (W,+). This subset is called the non-negative integers.

There are various elementary devices for suggesting the equivalence relation $\sim$ in $W \times W$ that generates the appropriate equivalance classes (elements of $W \times W/\sim$). Some devices involve diagrams in lattices, such as Diagram 1, where the elements in each diagonal subset form an equivalence class.

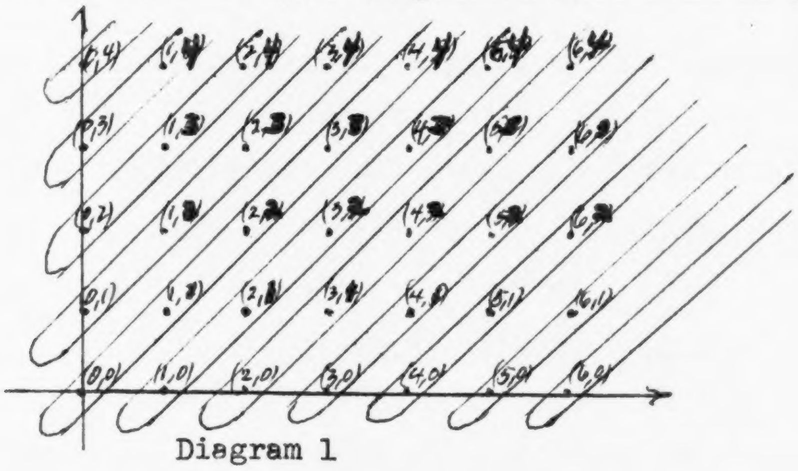

Diagram 1

Other devices involve arrows between number scales (esentially defining translations, such as in [4]). All these are devices to identify the classes generated by the relation $\sim$ given by

$$\forall (m,n),(m',n') \in W \times W \; [(m,n) \sim (m',n') \Leftrightarrow m+n'=n+m']$$

If we denote the class of all pairs equivalent to (a,b) by $(\overline{a,b})$, then

$$(\overline{a,b}) \text{ is called } \begin{cases} \text{a positive integer if } a>b \\ \text{the zero integer if } a=b \\ \text{a negative integer if } a<b \end{cases}$$

Addition in Z is the well defined operation $\oplus$ given by:

$$\forall\ (\overline{a,b}),(\overline{c,d}) \in Z, \; [(\overline{a,b}) \oplus (\overline{c,d}) = (\overline{a+c,b+d})]$$

Multiplication in Z is the well defined operation $\odot$ given by:

$$\forall (\overline{a,b}),(\overline{c,d}) \in Z \; [(\overline{a,b}) \odot (\overline{c,d}) = (\overline{ac+bd,\ ad+bc})]$$

All the ring properties and order properties of Z then follow from these definitions and the properties of $(W,+,\cdot)$.

4. Integer as an element of the set $(\{\wedge\} \times N) \cup W$.

Such an approach to Z involves a simple adjoining of new elements to the set W so that the resulting union admits an addition that agrees with addition in W and preserves the properties of addition in W, and also admits subtraction as an operation on the union.

Let N be the set of natural numbers $\{1,2,3,\ldots\}$ and define $Z = (\{\wedge\} \times N) \cup W$, where $\{\wedge\}$ is a singleton set. Denote the element $(\wedge,p)$ of $\{\wedge\} \times N$ by $\hat{p}$ With this definition of Z it can be proved that there exists an associative and commutative operation $\oplus$ on Z, with $0 \in W$ as identity, such that

a. $\forall m,n \in W \ [m+n = m \oplus n]$ ($\oplus$ preserves + in W)

b. $\forall a,b \in Z, \exists x \in Z \ [a \oplus x = b]$ (subtraction is an operation in Z)

c. $\forall a \in Z, \exists m,n \in W \ [n \oplus a = m]$.

Knowing that an appropriate definition of addition is possible, it is a simple matter to find such a definition in an elementary program.

The proof of the existence of an addition $\oplus$ on Z that satisfies the desired properties (a), (b), (c) is given in [5]. An outline of it is as follows: The operation $\boxplus$ on W x W given by

$$\forall (m,n),(m',n') \in W\times W \ [(m,n) \boxplus (m',n') = (m+m',n+n')]$$

is associative, commutative and has (0,0) as identity. Consider the functions

$$f: W\times W \to Z, \text{ where } \forall m,n,k \in W \ [f(m,m+k) = \hat{k} \text{ and } f(n+k,n)=k]$$

and

$$g: Z \to W\times W, \text{ where } \forall m \in W \ [g(m)=(m,0] \text{ and } \forall p \in N \ [g(\hat{p})= (0,p)].$$

Then the operation $\oplus$ on Z is defined in terms of $\boxplus$, f, g:

$$\forall a,b \in Z \ [a \oplus b = f(g(a) \boxplus g(b))].$$

In the process of proving that $\oplus$ has the desired properties we must show that

(1) $\forall m,n,d \in W \ [f(m,n) \boxplus (d,d) = f(m,n)]$

(2) $\forall (m,n) \in W\times W, \exists W\times W \ [gf(m,n) + (d,d) = (m,n)]$

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(3) $\forall (u,v),(m,n) \in W\times W \ [f(u,v) \oplus f(m,n) = f((u,v) \boxplus (m,n))]$

The results (1) and (2) come from the definitions of f and g. (3) follows from (1) and (2):

$$\begin{aligned} (u,v) \oplus f(m,n) &= f(gf(u,v) \boxplus gf(m,n)) \text{, by definition of } \oplus \\ &= f\big(gf(u,v) \boxplus (d,d) \boxplus gf(m,n) \boxplus (d',d')\big) \text{, by (1)} \\ &= f\big((u,v) \boxplus (m,n)\big) \text{, by (2), where} \end{aligned}$$

(d,d), (d',d') were chosen so that $gf(u,v)+(d,d) = (u,v)$ and $gf(m,n)+(d',d') = (m,n)$.

Then f is used to transfer additive properties of WxW to Z. Thus $\oplus$ is associative and commutative, $0 = f(n,n)$ is the identity, and each $a \in Z$ has an additive inverse $a' \in Z$, since for $a = f(m,n)$, let $a' = f(n,m)$, and then $a \oplus a' = f(m,n) \oplus f(n,m) = f\big((m,n) \boxplus (n,m)\big) = f(m+n,m+n) = 0$.

All the desired properties (a),(b),(c) then follow, and the existence of $\oplus$ is proved.

It can also be shown (again see [5], p.83) that the properties (a),(b),(c) of $\oplus$ on $Z \supset W$ determine Z uniquely up to a bijection. Thus any definitions of Z and addition on Z that satisfy the usual structural properties of Z are essentially equivalent. Thus in an elementary program in which the integers are defined by adjoining a new set to W, the definition of addition on Z can be suggested by appealing intuitively to any of the devices described above that involve translations or pseudo vectors. Addition in Z will be a binary operation with appropriate properties, while the computations of sums that addition assigns to pairs of integers can be carried out using any convenient device.

A similar proof shows that with Z defined as $(\{\wedge\} \times N) \cup W$ there exists an associative and commutative binary operation $\odot$ on Z, distributive over $\oplus$, with 1 as identity, such that

(d) $\forall m,n \in W, [m \cdot n = m \odot n]$, ($\odot$ preserves $\cdot$ in W).

The proof involves the binary operation $\boxdot$ on WxW, given by

$\forall (m,n),(m',n') \in W \times W\ [(m,n) \boxdot (m',n') = (mm'+nn', mn'+nm')]$, which is associative, commutative, distributive over $\boxplus$, and has (1,0) as identity. Then if $\odot$ on Z is defined by:

$$\forall a,b \in Z\ [a \odot b = f(g(a) \boxdot g(b))]$$

it can be shown that

$$\forall (u,v),(m,n) \in W \times W\ [f(u,v) \odot f(m,n) = f((u,v) \boxdot (m,n))].$$

Again f is used to transfer multiplicative properties of WxW to Z, namely associativity, commutativity and distributivity over $\oplus$. The identity is $1 = f(n+1,n)$.

Eg. $\hat{3} \oplus 2 = f(g(\hat{3}) \boxplus g(2)) = f((0,3) \boxplus (2,0)) = f(2,3) = f(2,2+1) = 1$

$\hat{3} \odot 2 = f(g(\hat{3}) \boxdot g(2)) = f(0,3) \odot (2,0)) = f(0,6) = f(0,0+6) = 6$

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B. Comparative analysis of the various approaches to the integers.

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Any analysis of the above approaches to the integers must be based on pedagogical grounds, since we have seen that each of the approaches leads to the same ultimate structure of $(Z,+,\cdot)$. We address the question, "What are the relative pedagogical merits of the various definitions of Z?"

Let us assume that a child has come to a point in his mathematics program where he has some notion concerning the whole numbers. To him a whole number might be a single symbol, such as "3", or a single point on the number line, or possibly an idea related to the concept of "all sets with three elements". He probably is most comfortable with these numbers when he is counting objects: that is, counting up through W, one by one.

When there is some reason for him to count down from some element of W and must stop at 0, it seems pedagogically most natural at this point to invent new words, or new labels for points not yet labeled on the number line, or new numbers, with which to continue to count down. The union of this set of new objects with W has unlimited capacity for counting up or down, and hence is an immediately available model for any situation in which such counting might occur. Indeed, such a model is often invented by children (for scoring in games) without prodding from teachers.

Compare this with the other approaches to integers, in which an integer is either a translation or a pseudo vector or an equivalence class of pairs of whole numbers. Each of these approaches requires of the child a level of maturity far in excess of that required for "counting down." Furthermore, each of these meanings of an integer describes an entirely new set of objects, so that W is not a subset of Z. Then an even higher level of maturity is required to understand that W is imbedded in Z in the sense that there is an isomorphism between W and a subset of Z.

Another pedagogical difficulty with the definition of an integer as a mapping involves the hierarchy of concept formations when learning the notion of a mapping. The first level of concept here is that of an assignment from one set to another. That is, a mapping "operates" on something, or assigns elements to something, so that the initial emphasis is on the elements in the domain and in the range of the mapping. A second level is that of composites of mappings, and at yet a higher level comes the concept of a mapping as an object or element of a set on which an operation is defined. (In fact, this latter level is so sophisticated that some college students never grasp it.)

Thus, if an integer is a mapping, the first level of concept formation requires the domain of the mapping be some familiar infinite set. But at this point the only familiar sets are W and possibly the set P of points of a line. We must reject W as domain, since for a translation $f:W \rightarrow W$, where $\forall n \in W, f(n) = n + m$, there is no inverse function defined on W. This leaves the set P as domain, which is satisfactory for the purpose. But then at a second level the child must accept a mapping of points of a line as an object which we call a number!

The same difficulties apply to the definition of an integer as a pseudo vector. For example, the Postman stories and Pebbles in a Bag stories are concerned with changes in amounts of pebbles or in changes in amounts of dollars.

The settings of the stories assume implicitly, for example, that there are always enough pebbles in the bag or dollars in the bank so that the indicated changes in amounts could be accomplished by removing pebbles or removing dollars. But for a fixed whole number n of pebbles in the bag we cannot describe a change of amount of "m less than" for m>n. Since the stories are not concerned with the amounts, only with changes in amounts, it is not clear how a child eventually identifies "n more than" with the whole number n, which is in conflict with his understanding of n as representative of an amount. Even more ellusive is the eventual identification of "n less than" with a negative integer, which then may represent an amount or state. Dienes in his work [6] is particularly careful to make clear the distinction between changes of state, which he describes as "difference-numbers," and states, which correspond to integers. The distinction is so basic to him that he apparently prefers to postpone the teaching of integers to a secondary program.

It might be argued that the natural sitations that motivate any approach to the integers are concerned with new entities associated with "above states" and "below states" and thus suggest new numbers +n and -n. But if we examine natural language we find that it is customary to say, for example, "My bank balance is $100", meaning "$100 in the black"; to say, "The temperature is 14°", meaning, "14° above 0 ": to say, "That was the year 1016", meaning the year 1016 AD". On the other hand, we are careful to say, "My bank balance is $100 in the red, or $100 overdrawn". "The temperature is 14° below zero"; "The elevation is 425 feet below sea level"; "That was the year 1016 BC". Thus, it seems to be part of our customs to describe "positive" quantities with counting numbers but to develop special words to indicate "negative" quantities. Thus, it seems most efficient to keep the set W and attach new numbers to it for new purposes, rather than to put W aside, introduce a new set of numbers, and then be faced quickly with the conceptual difficulty of identifying W with a subset of these new numbers.

C. The recommended approach to integers in the package program.

The above analysis suggests that the most natural and the simplest approach from the pedagogical standpoint is the one in which Z is the union of W and a new set needed for "counting down" from 0. This definition immediately responds to all the natural situations in which a child might cound down as well as up, such as gains and losses, above and below ground level, games with scores below as well as above zero, etc.

Then a definition of addition in Z is suggested by these same natural situations. This can be done by noticing that all these natural situations are simulated by compositions of translations of the number line. For example, let t be the unique translation of the Z number line defined by $0 \overset{t}{\to} 3$. Then 3 is the t-image of 0; $0 \overset{t}{\to} 3$. We might denote t by $\xrightarrow{\hat{3}\text{-slide}}$ or by any other symbol suggesting "counting down three units on the number line."

In the package program, sums of integers can then be computed using compositions of translations:

$0 \xrightarrow{\hat{3}\text{-slide}} 3$, $0 \xrightarrow{5\text{-slide}} 5$, $0 \xrightarrow{\hat{3}\text{-slide then } 5\text{-slide}} 2$, which suggests

$\hat{3} + 5 = 2$.

Notice the careful distinction between a $\hat{3}$-slide (a function) and the integer $\hat{3}$. One could be defined in terms of the other in the sense that $\hat{3}$ is the integer that is the image of 0 under the $\hat{3}$-slide, and the $\hat{3}$-slide is the (unique) translation of the number line that maps 0 onto $\hat{3}$.

The same distinction is made in the physical models of (Z,+). In translating back and forth between (Z,+) and a model, the integers relate to amounts or states or quantities, such as "$\hat{3}$ corresponds to 3 below zero", whereas the translations relate to changes in amounts or states, such as "$\hat{3}$-slide corresponds to a decrease of 3 in temperature".

The above description of a suggested development of addition in Z needs justification.

One view of addition in Z stems from the need for integers in order to solve equations such as $3 + x = 1$ or the desire to have an additive inverse or "opposite" for each whole number. The view we are justifying holds that the basic motivation for extending W to Z precedes even the additive structure of W. The set W is a Peano system, i.e., a set in which one can identify a first element (0) and in which each element has a successor. This structure of W, and only this structure, is exhibited when one displays W on the number line. The immediate motivation for extending W to Z now comes about because one would like to have a number precede 0; i.e., just as 1 is the successor of 0, one would like 0 to be the successor of some number say x. Then one would like a predecessor of x, and so on. One is, so to speak, "counting down" from 0. It is this structure of Z that is exhibited on the integer number line.

Z (or W) then does not come equipped with addition, but only with a "successor function" $\sigma: Z \rightarrow Z$. We suggest a definition of addition in Z which associate with each $x \in Z$ a translation σx of the number line such that $\sigma_x(0) = x$. The composite of two translations is again a translation, i.e., $\sigma_x \circ \sigma_y = \sigma_z$, and z is unique. It is this number z which we name "x + y". That this is reasonable for $x, y \in W$ can be seen by an argument involving counting, cardinality of sets, etc. But this is not the only way that a child might think about addition in Z. We see also that if $\sigma_x \circ \sigma_y = \sigma_z$ then $\sigma_y(x) = z$. This has a two-fold consequence: a) given an addition problem, $x + y = ?$, I may solve it by thinking either about $\sigma_x \circ \sigma_y = \sigma_?$ or $\sigma_y(x) = ?$, whichever I prefer: b) given a "word problem" which interprets naturally on the number line as $\sigma_x \circ \sigma_y = \sigma_?$ or $\sigma_y(x) = ?$, I may solve it by writing $x + y = z$, appropriately. Neither interpretation is more fundamental than the other. Each provides a device for computing the number z that is assigned to (x,y) by the binary operation of addition in Z. In practice, an addition slide rule would be a satisfactory device for performing this computation.

III. Suggested activities in the integer packages

A. Development of (Z, +, <)

1. Motivating and introducing the integers.
    a. Pose questions in various natural situations in which there are "changes of state" among quantities described as "above zero" or "below zero".

    For example:

    1) Picture of thermometer and questions about temperatures and changes in temperatures leading up to questions such as: If temperature rises 10° and then falls 20°, is it warmer or colder than before? How much warmer or colder? If temperature was 0°, what is temperature now? (Use natural language: temperature now is 10 degrees below zero.) Allow (in fact, encourage) counting up for warming and counting down for cooling.

    2) Picture of elevator shaft and same sort of questions about elevations and changes of elevations, leading to: If on the first trip you went up 7 floors and on second trip went down 9 floors, how many floors above or below original floor are you? What single elevator ride would

have taken you from original floor to final floor? If you started at ground floor, on what floor are you now? (Now on second floor below ground.) Again count up and count down.

3) Same kind of questions about profit and loss, leading to questions such as: On first business deal lost $20 and on second gained $10, how much richer or poorer now than at beginning? What single transaction would have had same effect? If I started with no money of my own, what is my final balance? (Now I have a $10 debt: I have to borrow $10 to satisfy the business deals.)

4) Time and change in time: number of chips and wins and losses in a poker game, etc.

5) Black and Red Army stories, Postman stories, Shuffle board games as in Book O, Chap 2, pp. 1-5

b. Tell story or play game in which there are "on points" and "off points". (Several possibilities like parchese, with moves made forward and backward along an irregular track or path from a starting position, each move determined by a spinner or by rolling two dice: in the latter case the moves would be made according to "on" or "off" commands on the tops of the dice and might suggest the possibility of "combining" the commands before making the move.) Then the language and format of the game can be reinterpreted in terms of the specific language of each of the situations described in a. above.

c. Questions about the game, such as: "3 off" on first spin and then "2 on" on second spin is same as "1 off" on a single spin, or "4 on" on one die and "10 off" on other die tells me to move "6 off".

d. Tell the same story or play the same game on the W number line (with points marked but not named on the other side of 0). Make starting point at 0 and let "3 on" mean move in the direction from 0 to 1 three units, and "3 off" mean move three units in the opposite direction. Play game until children sense a need for names for the points on which they place their checkers. Then name them "$\hat{1}$", "$\hat{2}$", ..., read "one-hat, two-hat,"... .

e. Relate the moves in the game to slides of the extended number line and name these slides:
"3 on" corresponds to slide named "3-slide": i.e.,

$$0 \xrightarrow{\text{3-slide}} 3,\quad 6 \xrightarrow{\text{3-slide}} 9, \ldots$$

"3 off" corresponds to slide named "$\hat{3}$-slide"; i.e.,

$$0 \xrightarrow{\hat{3}\text{-slide}} \hat{3} \quad 2 \xrightarrow{\hat{3}\text{-slide}} \hat{1}, \ldots$$

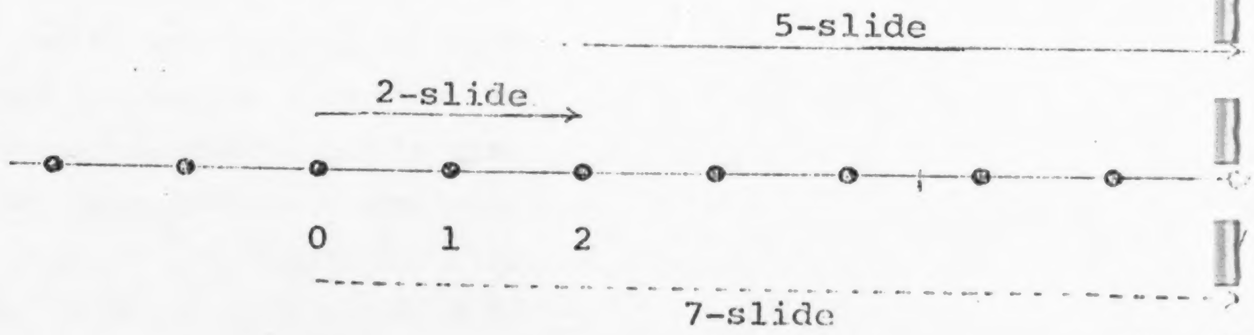
2. Addition in the integers.

a. Ask questions about composition of slides of the number line:
$\hat{4}$-slide and then $\hat{5}$-slide has same effect as a $\hat{9}$-slide
5-slide and then $\hat{5}$-slide has same effect as a 0-slide

b. Relate (or review) addition in W to slides of the number line:

$$0 \xrightarrow{\text{2-slide}} 2 \xrightarrow{\text{5-slide}} 7$$

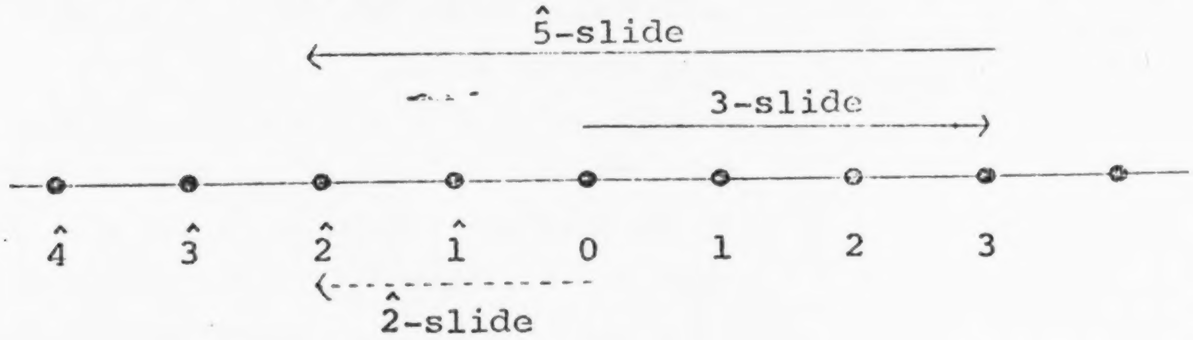
i.e., 2-slide and then 5-slide has same effect as 7-slide. A corresponding true statement is 2 + 5 = 7, so that composition of slides is related to addition of whole numbers.

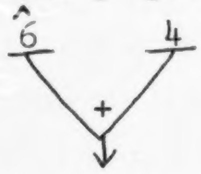


c. More generally, composition of slides of the number line suggest how to add any whole numbers or hat numbers:

$0 \xrightarrow{3\text{-slide}} 3 \xrightarrow{\hat{5}\text{-slide}} \hat{2}$

i.e, 3-slide and then $\hat{5}$-slide has same effect as $\hat{2}$-slide, suggesting the $3 + \hat{5} = \hat{2}$ addition statement.



d. Practice this arithmetic, using open sentences ($\square + 3 = \hat{4}$), diagrams , etc.

e. Give name "integers" to set of all whole numbers and hat numbers.

f. Build an integer addition slide rule based on the composition of slides of the number line. Take the results of such composition (as done mechanically by the slide rule) as the meaning of addition of integers.

3. Applications and concepts

a. Physical models: temperature, elevation, business status, etc. In each model make an assignment of the integers, such as

2 feet below sea level ⟶ $\hat{2}$

5 degrees above zero ⟶ 5, etc.,

and an assignment of the slides, such as

3 feet decrease in elevation ⟶ $\hat{3}$-slide

4 degree increase in temp. ⟶ 4-slide, etc.

Then relate addition of integers to situations in each model.

In the first hour of hiking John increased his elevation 25 feet; in the second hour he decreased his elevation 32 feet. At that time, how many feet above or below his initial position was he? Then, since 25-slide and then $\widehat{32}$-slide is a $\hat{7}$ slide, he is 7 feet below his initial position. Let his initial position be 0. Now his position is $\hat{7}$, or 7 feet below the 0 position. Draw graphs of some of these experiences -- time vs. position. Also let children read information from graphs. Write statements relating to situations. Also, let children describe situations relating to given statements.

b. Let the concentration now be on additive inverses, after it is clear that 0 is still the neutral element for addition in the integers. Emphasis on the fact that for each integer there is an integer which is its inverse in the sense that their sum is zero (the composition of their slides is the 0-slide, the do-nothing slide).

They can also be called "opposites" since they are opposite each other on the number line with respect to 0. What is the additive inverse or opposite of $\hat{7}$? (I.e., what is the solution of $\hat{7} + \square = 0$?) Discover that every integer has one and only one additive inverse. (This might be done in a game setting and then with diagrams.)

c. Let skill in addition come in any way -- from slides, or in any convenient model -- but preferably by slides. Hopefully we can get the children to think along these lines: When adding a negative integer and a positive integer, we "hunt" for zero in the sense of first sliding to 0 and then on to the other number (because 0 is the most convenient reference point). This is precisely what automatically happens when we use an addition slide rule. For example,

$$\hat{4} + 7 = \hat{4} + (4 + 3) = (\underbrace{\hat{4} + 4}) + 3 = 0 + 3 = 3.$$

hunting for opposite of $\hat{4}$, or
counting up from $\hat{4}$ to 0.

More tables, diagrams, hook-ups, open sentences, games for skills, all hinting at properties of (Z,+). Many of the devices used for (W,+), such as ordered pair games and ~~*~~#games, can be revived for (Z,+).

d. Order in the integers.

Review order in W on the number line. Then observe that in natural models a statement such as "10 feet elevation is $\left\{\begin{matrix}\text{below}\\\text{lower}\end{matrix}\right\}$ than 12 feet elevation" is related to the order statement "10 < 12." Extend to order in Z by obtaining statements such as $\hat{7} < \hat{4}$, $\hat{3} < 1$ by similarly appealing to the models: 7 feet below sea level is $\left\{\begin{matrix}\text{below}\\\text{lower than}\end{matrix}\right\}$ 4 feet below sea level; 3 degrees below zero is colder than 1 degree above zero. Thus make agreement on the convention that for a, b in Z, a < b iff on the number line a is on the same side of b as 0 is of 1. Using translations, children can see that a < b iff there is a positive slide taking a to b; i.e., a < b iff there is a natural number c such that a + b = b. This could then serve as the definition of < in Z.

e. Games and extensions

Besides the many possibilities of operational checkers games for ordered pairs of integers, including * games, there are interesting games to be played on the plane coordinatized with Z x Z. The coordinatization can be done with tic-tac-toe games and Battleship games, as extensions of these games played on W x W. Then mappings of Z x Z provide games of "find my point". Eg. If → → ↓ ↓ ↓ takes my point to (2,3), where am I? Or, I started at (2,1). To what point did ↓ ↓ ← take me? Here the notation → → ↓ ↓ ↓ for example, indicates a movement two units to the right and three units down, i.e., (x,y) → → ↓ ↓ ↓ (x + 2, y + $\hat{3}$). Composites of these mappings can be incorporated in simple stories or games, thus beginning the notion of vector addition in Z x Z.

Open sentences in Z x Z and their graphs can also be touched lightly after the plane has been coordinatized with Z x Z.

B. Notation

When chosing a symbol to represent a negative integer we must consider i) the needs of the short range conceptual development and ii) the long range necessity of ultimately conforming to the accepted symbol. For the short range it is certainly desirable to avoid the symbolic and conceptual confusions inherent in the standard symbol "-3" which has various meanings in various contexts.

It can mean:

1) "-3" is a pair of symbols, the "-" standing for the opposite mapping in Z and the "3" for the element in Z on which "-" is operating. In this context, "-3" is functional notation; if O: $Z \rightarrow Z$ is the opposite mapping, then "-3" is an abbreviation for "O(3)."

2) "-3" is a single symbol, with the "-" and "3" inseparable and forming one heiroglyphic, standing for the number in Z which is the image O(3) of 3 under the opposite mapping, or what is equivalent, the negative integer that is "3 below zero."

3) "-3" is a functional notation for the unary operation f_3 of subtracting 3; for any $a \in Z$, $f_3(a) = a - 3$, so that for example, "5 -3" is an abbreviation for "$f_3(5)$." Ultimately, each student who uses mathematics will learn to identify the meaning of "-3" from its context. But for the first introduction to integers it is essential that a special non-standard symbol be used for a negative integer so that only the meaning 2) above is carried by this symbol. Among the possible symbols considered were $^{-}3$, $\overline{3}$, $\hat{3}$. The first two still carry a dash, even though raised, which might suggest subtraction. Thus the third was adopted and read "hat 3." Thus "$\hat{3}$" is one symbol, and although it is derived from the standard name of three, we do not consider the "hat" to be a separate operation symbol. It is then meaningless to write $\widehat{5-2}$ or $\widehat{(\hat{3})}$ or $\hat{n}$, each of which would denote the "hat" as an operation. It is recommended that this "hat" notation be used for the negative integers throughout the primary program and be dropped only after the standard symbol has been introduced for the opposite mapping in Z and the students are completely at home in the additive structure of Z, including subtraction in Z. This means that the non-standard notation will be continued at least until the first year of the secondary program.

C. Subtraction and multiplication in Z.

1. Motivation for subtraction

At the beginning of the development of the integers, subtraction in Z should be somewhat soft-pedaled, but eventually it can act as an excellent setting in which again to suggest that one can view subtraction alternately in terms of addition. This is the case because in natural models of (Z,+) we are almost forced to state all problems in language that suggests addition; e.g., "How much time elapsed from the year 612 BC to the year 1969?" could be possibly distorted into a subtraction problem but would almost certainly be treated as an addition problem. It is thus almost certain that appeals to natural models will not suffice to motivate subtraction in Z for some children. Accordingly, we must wait until they have enough experience with (Z,+) so that questions of mathematical structure alone will provide enough motivation for development of

$a - b = a +$ (additive inverse of b).

There would be no harm done if this does not happen until the end of the primary program or beginning of secondary program. In the meanwhile, it is enough for children to be able to find a unique solution of $a + \square = b$ for any a,b in Z whenever such an open sentence is suggested in a natural model or in a game situation.

We point out the dangers of introducing artificial devices designed to convince children that subtracting 3 reverses things in some way or other and actually adds 3. This would imply some magical "change of sign" that should be avoided at all costs. If a child makes up rules of his own, that is another matter.

2. Introduction of subtraction in Z (not until grade 6 or 7)

Early in the program children will learn to solve open sentences in Z such as $a + \square = b$. If it is thought desirable to introduce the operation of subtraction late in the primary program, it might be done as follows:

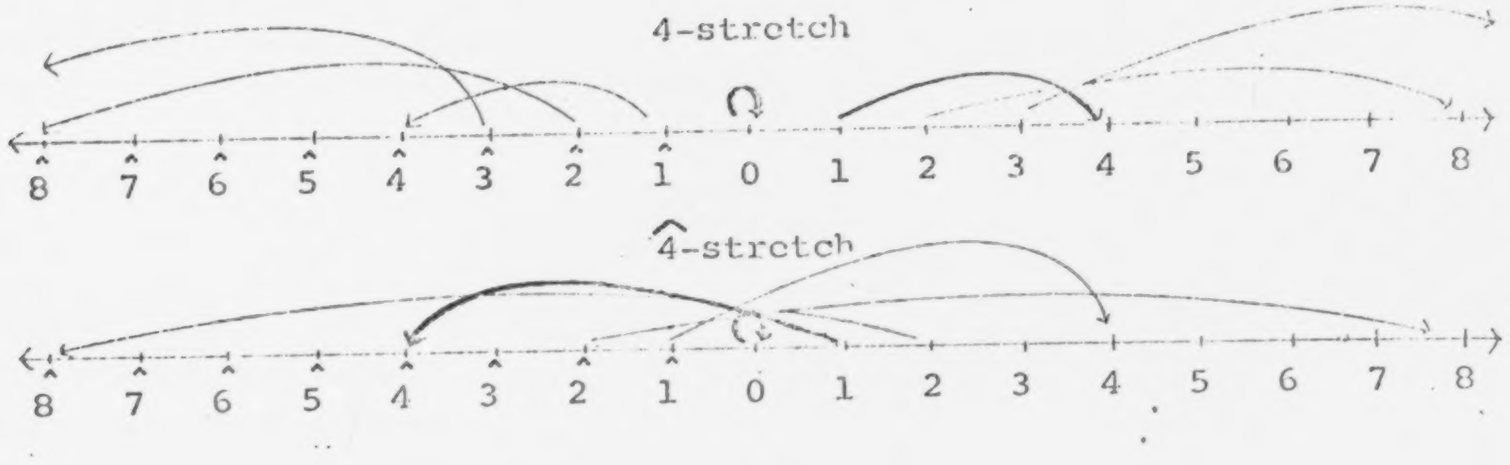
a. Reinforce the definition of $a - b$ in W, if this number exists in W, as the unique solution of $a = \square + b$. Solve such sentences in Z. Through games and diagrams decide that every such sentence has a solution in Z and only one.

b. Compare the solution of $a = \square + b$ with the solution of $a +$ (add inverse of b) $= \square$, but do not force the recognition that $a - b = a +$ (add inverse of b). One of many possible devices for leading children to this recognition is the analogy of a scale balance in which positive integers and corresponding negative integers "balance" each other. Then "subtracting an integer a from one side" of fulcrum to bring scale into balance is same as "adding opposite of a to other side." Here we must decide whether to introduce a symbol for the image of an element of Z under the "opposite" mapping now or postpone until later. If O: $Z \rightarrow Z$ is the "opposite" mapping (reflection of the Z-number line in the line through 0 and perpendicular to the number line) we could experiment with the symbol ^{-}b, where $O(b) = {}^{-}b$ for $b \in Z$. If this seems to be easily understood, we might begin to drop the hats and write $^{-}4 = \hat{4}$. (It is recommended that this be postponed until the beginning of the secondary program).

3. Motivation for multiplication.

Unlike (Z,+), which is easily related to everyday experiences, the system (Z,•) does not seem to have any sensible physical or natural models that belong to the realm of experiences of a young child. We could concoct such experiences, but who has found any that are not contrived? It seems best then to postpone development of (Z,•) for a given child until he shows enough curiosity about the abstract structure of Z to want to explore a possible meaning of multiplication in Z that is consistent with multiplication in W and has the properties we have come to accept as desirable. For some children, this might come immediately after some experience with (Z,+). For others, it might never come in the primary program. For the latter, there is no harm done because arithmetic as an everyday tool does not need to include multiplication in Z. For those in between, some may need to wait until a secondary program and others may be ready near the end of the elementary program.

When multiplication is introduced we must again avoid any device that contrives to suggest "rules of signs". What we might emphasize is the role of mappings, i.e., stretchers, such as given below, in the development of a device for computing the product of two integers (and thus giving a definition of the binary operation).


4-stretch
$\hat{8}$ $\hat{7}$ $\hat{6}$ $\hat{5}$ $\hat{4}$ $\hat{3}$ $\hat{2}$ $\hat{1}$ 0 1 2 3 4 5 6 7 8
$\hat{4}$-stretch
$\hat{8}$ $\hat{7}$ $\hat{6}$ $\hat{5}$ $\hat{4}$ $\hat{3}$ $\hat{2}$ $\hat{1}$ 0 1 2 3 4 5 6 7 8

an n-stretch is defined geometrically by the mapping $\mu_n(0) = 0$, $\mu_n(1) = n$, for each $n \in Z$.

On the other hand, we might rely on patterns of products in W and their apparent extensions to suggest a definition of products in Z, as described in 4. below.

We recommend that multiplication in Z be initially introduced no earlier than the sixth year and that it be motivated by following familiar patterns of products in W. Then, if the rationals are introduced in terms of stretchers and shrinkers in the package program, the same approach can be used for multiplication of integers.

4. Introduction of multiplication in Z (not until grade 6 or 7)
   1) Patterns in skip counting.

      Review skip counting in W, and then learn to skip count by 2's, by 3's, etc. by successive compositions of 2-slides, 3-slides, etc., making use of story situations:

      By 3's: $\ldots \hat{9},\hat{6},\hat{3},0,3,6,9, \ldots$

      By $\hat{3}$'s: $\ldots 9,6,3,0,\hat{3},\hat{6},\hat{9}, \ldots$

      (By this time in the program, students might be familiar with sets of multiples of whole numbers. If so, the extension can now be made to sets of multiples of integers.)

Then look at these patterns: [7]

$3 \cdot 3 = 3 + 3 + 3 = 9$

$3 \cdot 2 = 1 + 2 + 2 = 6$

$3 \cdot 1 = 1 + 1 + 1 = 3$

$3 \cdot 0 = 0 + 0 + 0 = 0$

$3 \cdot \hat{1} = \underline{\hspace{3cm}} = \underline{\hspace{0.3cm}}$

$3 \cdot \hat{2} = \underline{\hspace{3cm}} = \underline{\hspace{0.3cm}}$

$3 \cdot \hat{3} = \underline{\hspace{3cm}} = \underline{\hspace{0.3cm}}$

It would be natural to continue the patterns as $3 \cdot \hat{1} = \hat{1} + \hat{1} + \hat{1} = \hat{3}$, etc.

Next look at these patterns:

$3 \cdot 3 = 9$

$2 \cdot 3 = 6$

$1 \cdot 3 = 3$

$0 \cdot 3 = 0$

$\hat{1} \cdot 3 = \underline{\hspace{0.3cm}}$

$\hat{2} \cdot 3 = \underline{\hspace{0.3cm}}$

$\hat{3} \cdot 3 = \underline{\hspace{0.3cm}}$

It would be natural to notice the skip counting by 3's in the right column and continue it to obtain $\hat{1} \cdot 3 = \hat{3}$, $\hat{2} \cdot 3 = \hat{6}$, et

This suggests that the product of a whole number and a hat number is a hat number.

Then look at these patterns:

$$\left.\begin{array}{l} \hat{3} \cdot 3 = 3 \cdot \hat{3} = \hat{9} \\ \hat{3} \cdot 2 = 2 \cdot \hat{3} = \hat{6} \\ \hat{3} \cdot 1 = 1 \cdot \hat{3} = \hat{3} \\ \hat{3} \cdot 0 = 0 \cdot \hat{3} = 0 \end{array}\right\} \longleftarrow \text{From above suggestion.}$$

$\hat{3} \cdot \hat{1} = \hat{1} \cdot \hat{3} = \underline{\hspace{0.3cm}}$

$\hat{3} \cdot \hat{2} = \hat{2} \cdot \hat{3} = \underline{\hspace{0.3cm}}$

$\hat{3} \cdot \hat{3} = \hat{3} \cdot \hat{3} = \underline{\hspace{0.3cm}}$

Now it is natural to observe the skip counting by 3's in the right column and continue to obtain $\hat{1} \cdot \hat{3} = 3$, $\hat{2} \cdot \hat{3} = 6$, etc.

The suggestion is that the product of two hat numbers is a whole number.

It should be pointed out that the acceptance of the extensions of these patterns of products as a definition of multiplication in Z is equivalent to a requirement that the multiplication in Z be associative, commutative, and distributive over +, have 1 as identity, and preserve multiplication in W.

Let $a,b,c \in W$ and $a \cdot b = c$.

Then

$a \odot b = a \cdot b$.

Since

$c = a + a + \ldots a$ (b terms)

we have

$\hat{c} = \hat{a} + \hat{a} + .. + \hat{a}$ (b terms) .

Then

$$\begin{aligned} b \odot \hat{a} &= \hat{a} \odot b = \hat{a} \odot (1 + 1 \ldots + 1) \\ &= \hat{a} \odot 1 + \hat{a} \odot 1 + \ldots + \hat{a} \odot 1 \\ &= \hat{a} + \hat{a} + \ldots + \hat{a} \\ &= \hat{c} \end{aligned}$$

In particular,

$b \odot \hat{1} = \hat{b}$ and $\hat{a} \odot 0 = 0$.

Then

$$\begin{aligned} 0 = \hat{a} \odot 0 &= \hat{a} \odot (b + \hat{b}) \\ &= \hat{a} \odot b + \hat{a} \odot \hat{b} \\ &= \hat{c} + \hat{a} \odot \hat{b} \\ \Rightarrow \hat{a} \odot \hat{b} &= c, \end{aligned}$$

since c is the only solution of $0 = \hat{c} + x$.

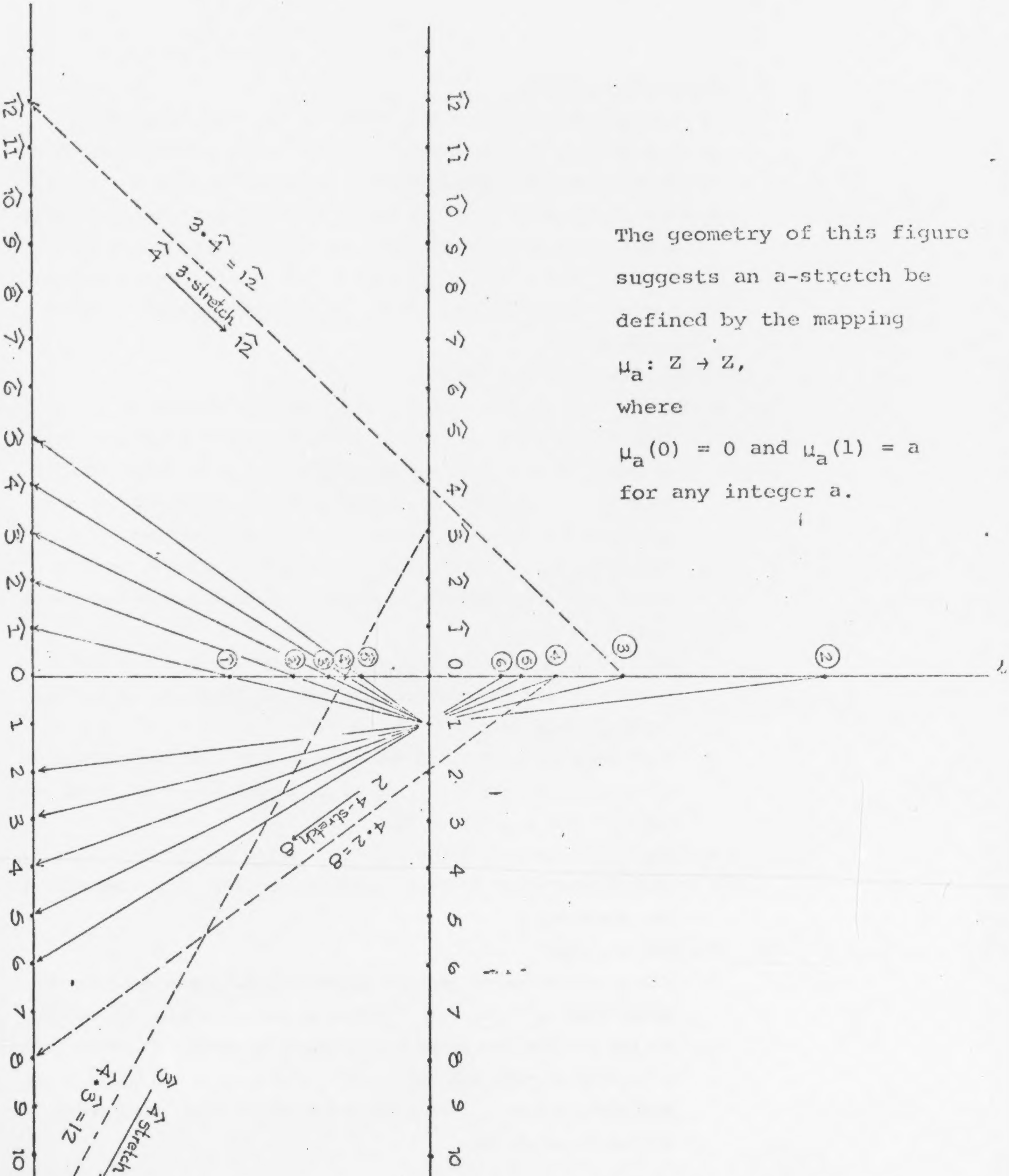
In this sense, the resulting definition of $\odot$ in Z is forced on us, given the structure of $(Z,+)$ and requiring the desired structure of $(Z,\odot)$.

2) Patterns in stretching maps of the number line.
Through story or game introduce stretching mappings of the number line, such as:
3 stretch : $0 \to 0$, $1 \to 3$, $2 \to 6$, $3 \to 9$, ...
In general, $a \xrightarrow{\text{3-stretch}} 3 \cdot a$ for $a \in W$.
Just as a device (addition slide rule) supplies sums in Z, we can supply a graphical device for applying stretching mappings, and hence for obtaining products in Z. Note that an n-stretch takes 0 to 0 and takes 1 to n, for any whole number n. On a diagram such as Diagram 2 we can connect the

1 of the top number line to n (image) on the bottom line and extend this line to cut line ℓ at $\textcircled{n}$.

This establishes n as a scale factor on the bottom number line.

To find the image of any whole number a under the n-stretch, we draw the line through $\textcircled{n}$ and the point for a on the top line, extend it and it hits the bottom line at the image, n•a. Now it is natural to extend this geometric scheme to an $\hat{n}$-stretch, for any whole number n; i.e., a mapping that takes 0 to 0 and takes 1 to $\hat{n}$. Again connect 1 on top line to $\hat{n}$ on bottom, and find point $\textcircled{\hat{n}}$ where this line cuts line ℓ. Etc. Such a graphical device could be used as a nomogram to determine products in Z. As such, it would be a "stretching machine" to determine images of stretch mappings. Its use in the package program would depend on the extent to which stretching operators are employed in the introduction to the rational numbers.


3·4̂ = 12̂
4̂ 3-stretch 12̂
2 4-stretch 8
4·2=8
3̌ 4̌ stretch 12
4̌·3̌=12
12̂ 11̂ 10̂ 9̂ 8̂ 7̂ 6̂ 5̂ 4̂ 3̂ 2̂ 1̂ 0 1 2 3 4 5 6 7 8 9 10 11 12
①②③④⑤⑥

The geometry of this figure suggests an a-stretch be defined by the mapping

$\mu_a: Z \to Z$,

where

$\mu_a(0) = 0$ and $\mu_a(1) = a$

for any integer a.

3. Arithmetic in $(Z, \cdot)$

If multiplication in Z is introduced in the sixth year of the package program, some arithmetic skills can be developed in the usual ways: solving open sentences, drawing flow diagrms (always hinting at the ring structure of Z), playing operational checkers games with ordered pairs of integers under multiplication. It is recommended that formal statements of the multiplicative properties of Z be postponed until the first year of the secondary program. (See Book O, Chap. 2).

D. Prerequisites and parallel packages

1. Prerequisite skills and concepts prior to introduction of Z
   - a. Understanding of addition as an operation on W and some skills in addition in W (not necessarily including an algorithm for addition), with some understanding of the commutativity and associativity of addition and the role of 0 as identity.
   - b. Skills in using number line, coordinatized with W, to skip count, and understanding of order in W relative to the number line.
   - c. Ability to solve open sentences in W of the form $a + \square = b$, $a + b = \square$ and to construct and read diagrams and tables involving operations in W
   - d. Understanding of a slide motion of a line (not necessarily a horizontal line) and ability to determine images of given points under a given slide motion
   - e. Ability to perform compositions of slide motions of a line, and given a slide to find another such that their composite is the identity.
2. Parallel packages
   - a. Coordinatization of part of plane with WxW (prerequisite for later work in integers). Such a package could be introduced by the tic-tac-toe games and Battleship games. It could include solutions of open sentences such as $\square + \Delta = 6$ and $\square \cdot \Delta = 12$, and their graphs. Such experiences could then be extended to similar ones in ZxZ.

b. Clock arithmetic (not a prerequisite for any part of the integer packages)
   The sets Z_5, Z_6 (or any cyclic group isomorphic to Z_5 or Z_6, such as the set of rotation~~s~~al symmetries of a pentagon or hexagon about its center, under composition). Tables of operations on these sets and open sentences with replacements from these sets might suggest properties of identity and inverse elements parallel to those in (Z,+). Most of this sort of package could be presented in settings of games.

c. Stretcher and shrinker operators on W (not a prerequisite)
   Such a package might find its way into an introduction of rational numbers. If so, it might be a useful resource when multiplication in Z is developed.

IV. Short and long term goals

A. Long term goals (by end of total program) for all students.

1. Skill of constructing and labelling an integer number line; concept of set of integers as an extension of the set of whole numbers and as a set admitting counting down as well as counting up.
2. Skills in addition in Z (without an algorithm)
3. Skill in recognizing physical models of (Z,+) and skill in shifting back and forth between (Z,+) and any of the models in order to solve simple problems in the models.

B. Short term goal (by end of primary program) for top level students.
Development of skills and concepts with respect to the system (Z,+,•,<) comparable to that given in Book 0, Chap. 2 of the E.M. series.
Note: There is a wide spectrum of goals between A and B above that will be dictated by individual students' abilities and interests. It is not even clear that all students will be able to attain the goals of A in the primary program, although hopefully they will in the total program.

C. Long term goals (by end of total program) for top level students.
Understanding of the role of Z in mathematics both structurally and computationally, as described in IA; understanding of the construction of Z from W and the characterization of Z as a subsystem of R.
Development of concepts comparable with ~~that~~ those given in Books 3-8, 11 in the E.M. series.

V Bibliography

1. MacLane, Mapping as a Basic Mathematical Concept, 12th Scientific Seminar of the Air Force
2. Papy, Mathematique Moderne I, Herman (?) Chap. 2
3. Davis, Discovery in Math., Chapter 12, Explorations in Math., Chaps. 5 &
4. Freund, MNV 18 (1965/66) pp. 401-406.
5. MacLane. Birkhoff, Algebra, MacMillan, 1967, p. 43
6. Dienes, Modern Mathematics for Young Children, Herder & Herder, 1966.
7. Entebble Primary Series, Primary 5, Teachers' Book

ACTIVITY SEQUENCE OUTLINE
ON MOTION GEOMETRY (MG)

The overall mathematical goals are:

(1) Establish an intuitive notion of congruence in the plane;

(2) Introduce three Euclidean transformations: line reflection, translation, and rotation;

(3) Suggest a characterization of the three transformations according to their properties, such as:
    (a) number of fixed points,
    (b) orientation-reversing, or not;

(4) Consider composition of these three transformations (a good first exposure to Cartan-Dieudonné theorem in 2-space).

Activity 1: MG03

Introduction to the notion of match or matching (corresponds to congruent), and to one use of tracing paper. In context, the word "figure" will be used in ways that will naturally suggest the idea that a figure is a set of points.

At the end of this activity the student should be able to

(1) given two figures, answer correctly the question, "Do the figures match?"

(2) given many figures on a page and a model figure (or figures), show all those figures which match the model.

Activity 2: MG06

Video-tape demonstration of how to use tracing paper and carbon paper in drawing figures match a given figure. There should be some emphasis on careful tracing and on the use of masking tape to keep paper from moving. Sample language which will be used: "Match original figure", "matching image", etc.

At the end of this activity the student should be able to

(1) given a figure, draw a matching figure (called a matching image);

(2) given a figure and part of a matching figure, complete the matching image so that the part given is in this image.

Activity 3: MG07

Practice on drawing matching figures and recognizing when two figures are matching. This activity reviews ideas from MG03 and MG06, but begins to suggest special kinds of matching images.

Activity 4: MG09

Suggest that congruence is an equivalence relation. Begin using grid paper as background and check for congruence without tracing.

Activity 5: MG12

Begin emphasis on point mappings using

$$a \rightarrow b \quad \text{(a goes to b)}$$

for "the image of point a is point b". This notation will be especially used in giving directions. We also begin to suggest that the images of three non-collinear points are sufficient to determine a matching image of any given figure.

Pages 5, 6, and 10 are good examples of what the student should be able to do at the end of this activity.

Activity 6: MG15

A video tape in which the notion of a slide mapping (translation) is introduced. The student is given a method for drawing matching images under a slide mapping. The mapping is determined by a slide arrow and the image under this mapping is called slide image. Sufficient stress is put on finding slide images of individual points, suggesting that a slide mapping can be determined by the image of one point (knowing the mapping is a slide). Students will be encouraged to use a straight edge to draw slide arrows.

At the end of this activity the student should be able to draw a slide image of some figure given a slide arrow.

Activity 7: MG16

Exercises in drawing slide images and recognizing that a figure is, or is not, the slide image of some original figure, given a slide mapping.

At the end of this activity the students should be able to

(1) given an original figure and a slide arrow, pick out one among many figures as the slide image;

(2) find and label slide images of individual points;

(3) given an original figure and its slide image, draw an appropriate slide arrow;

(4) given an original figure, a slide image, and many possible arrows for the slide mapping, pick out all those which are appropriate;

(5) given two matching figures and a slide arrow answer correctly the question "Using this slide arrow, is figure x the slide image of figure y?"

Activity 8: MG17

The emphasis in this activity is on some properties of slide mappings, for instance:

(1) knowing that a mapping is a slide, it can be determined by the image of one point;

(2) a slide mapping followed by a slide mapping is again a slide mapping;

(3) the inverse of a slide mapping is a slide mapping;

(4) a slide mapping has no fixed points.

At the end of this activity students should be able to

(1) given an original figure and part of its slide

image, draw the appropriate slide arrow and complete the slide image;

(2) given an original figure and the slide image of one point in the plane, draw an appropriate slide arrow and draw the slide image of the original figure.

Activity 9: MG22

A video-tape introduction to the notion of a fold mapping (reflection). The tape begins by suggesting, although not explicitly, that one major difference between fold mappings and slide mappings is that the former is orientation-reversing, whereas the latter is orientation-preserving. A method is given for drawing the fold image of some original figure given a fold line.

At the end of this activity the student should be able to

(1) given an original figure and a fold line, draw the image;

(2) given an original figure and a fold line, pick out one among many figures as the fold image.

Activity 10: MG23

A video tape in which some special fold mapping problems are discussed. This includes finding and drawing the fold line given an original figure and part of its fold image; determining for two given figures whether one is a fold image of the other and, if yes, drawing the fold line; drawing fold images of figures which intersect the fold line.

Activity 11: MG24

Continuation of work with folds where students can practice the techniques learned in MG22 and MG23. There should be a mixture of fold and slide mapping exercises, for comparison purposes. Also, this activity begins to suggest some properties of fold mappings, e.g., a fold mapping has fixed points, namely those on the fold line; a fold mapping followed by a fold mapping is not a fold mapping; and the inverse of a fold mapping is a fold mapping.

At the end of this activity the student should be able to

(1) given an original figure and a matching image,

tell whether the image is, or is not, a fold image. If it is a fold image, the student should be able to draw the fold line;

(2) given two figures and a fold line, answer correctly the question "Is figure A the fold image of figure B for this fold mapping?";

(3) given an original figure and part of its fold image, draw the fold line and complete the fold image.

Activity 12: MG27

A video-tape introduction to the notion of a turn mapping (rotation). A turn center and a turn arrow will be used to describe a turn. The students will learn a method for drawing turn images. They will also be introduced to a compass and how it can be used to draw turn arrows.

Activity 13: MG28

Practice in drawing and in recognizing turn images. Begin suggesting some properties of turns, such as that a turn has one fixed point (the turn center).

At the end of this activity the student should be able to

(1) given an original figure, a turn center, and a turn arrow, draw the turn image;
(2) given an original figure, a turn center, and a turn arrow, pick out the turn image from among several figures;
(3) given an original figure, turn center, and turn image, draw a turn arrow for the turn mapping.

Activity 14: MG31

More exercises on turn mapping with a comparison between this and the fold and slide mappings. The notion of fixed points will be carefully looked at in terms of all three mappings. Begin looking at patterns in the plane which have rotational, translational, and reflective symmetry.

Activity 15: MG34

Compositions of the three mappings. Some properties of composition will be suggested (as was done earlier with respect to individual mappings), such as that composition is not commutative.

Activity 16: MG37

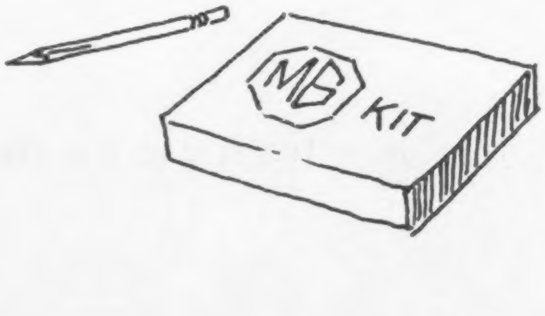
Special compositions hinting at the Cartan-Dieudonné Theorem in 2-space; that is, any rigid motion in the plane can be realized as a composition of three or fewer reflections.

Name ____________________

DRAWING
MATCHING FIGURES
II

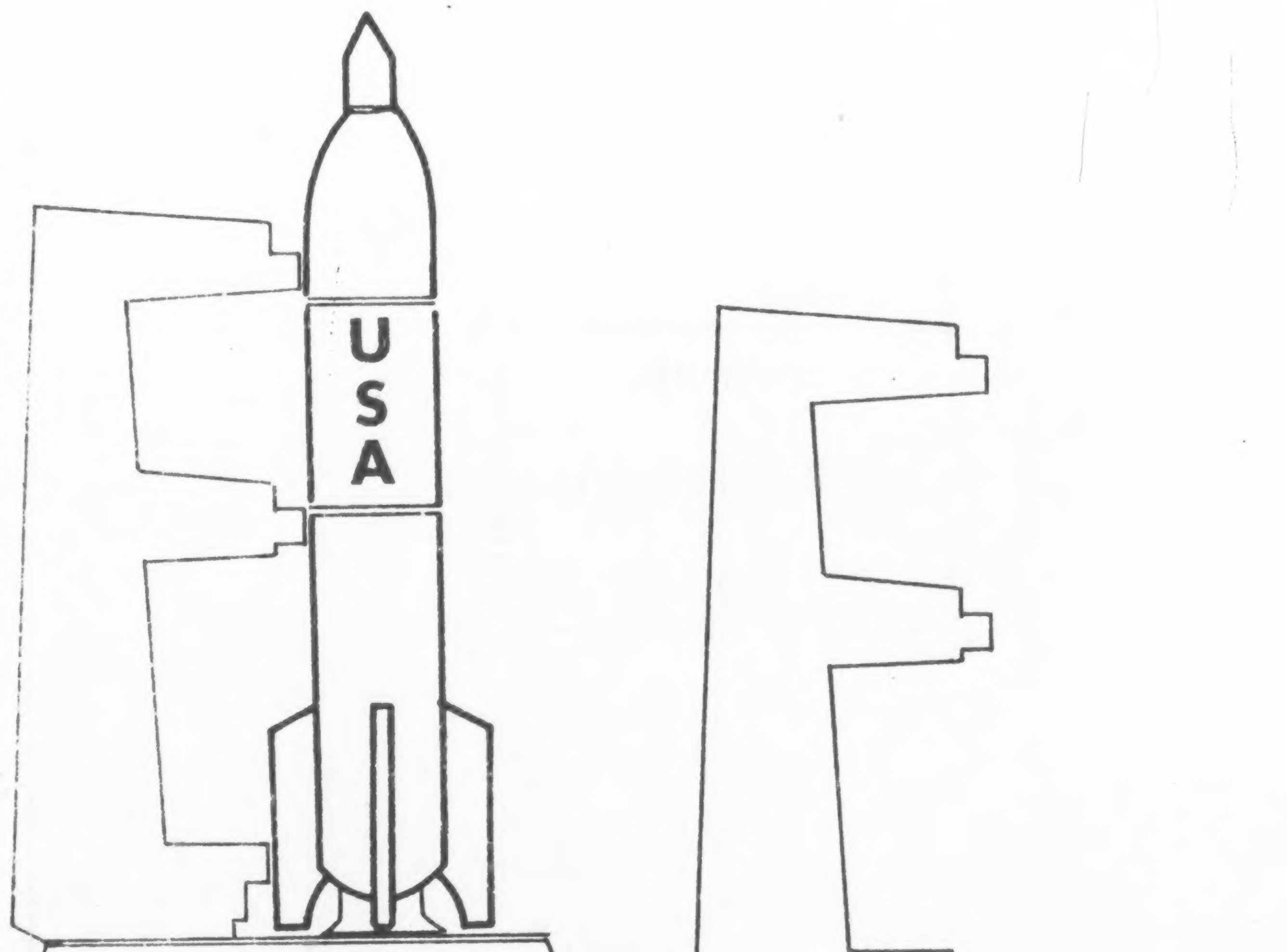
For this activity you will need:

1) A pencil
2) An MG kit



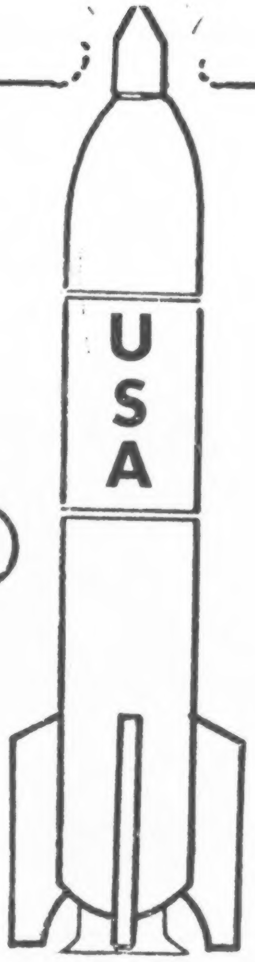
MG07

Draw a matching rocket on lauching pad 13.


USA
Launching Pad 12
Launching Pad 13

Save your tracing of the rocket for page 2.

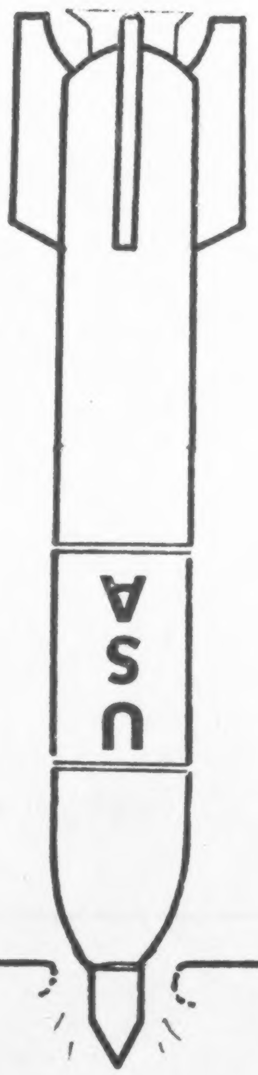
Your tracing from page 1 should match figure A. Check it.



Ⓐ

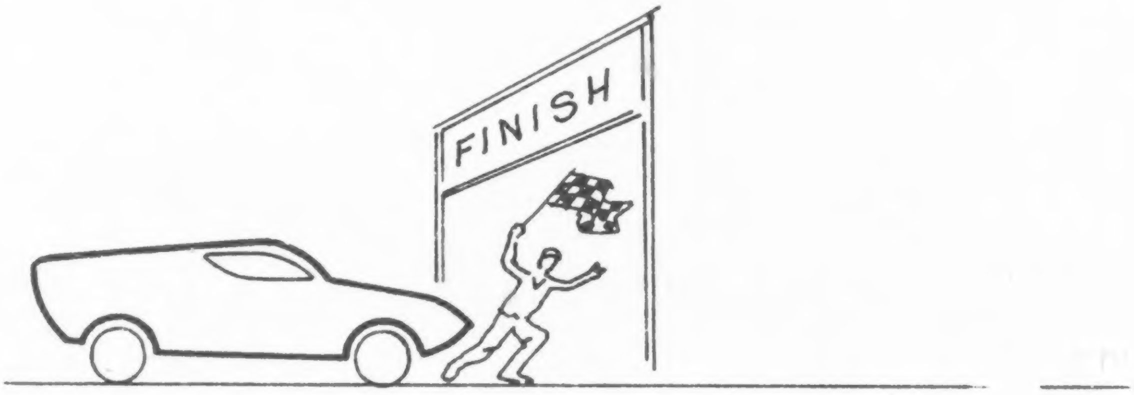
Do figure A and figure B match ? ☐ Yes ☐ No

Use your tracing from page 1 to be certain.



Ⓑ

Draw a matching car which has already crossed the finish line. Use tracing paper.



Save your tracing of the car for page 4.

Circle all figures which match figure A.

Cross out all figures which do not match figure A.

Use your tracing from page 3.

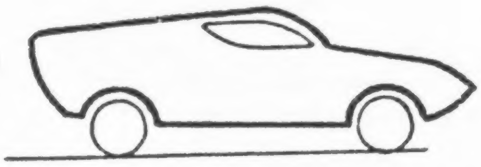
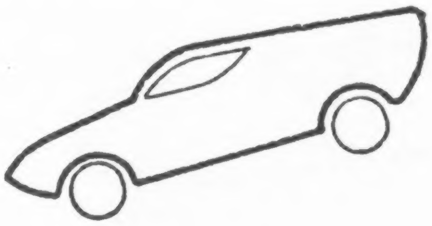
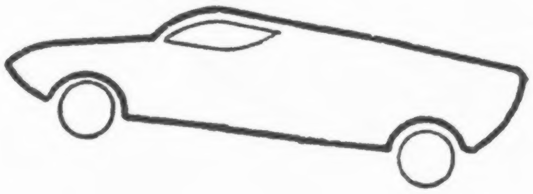
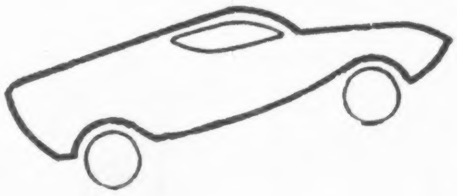
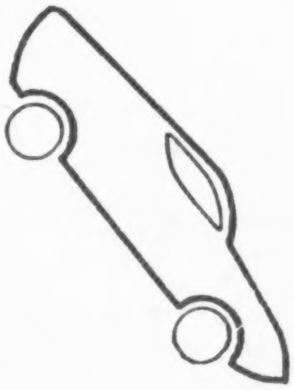


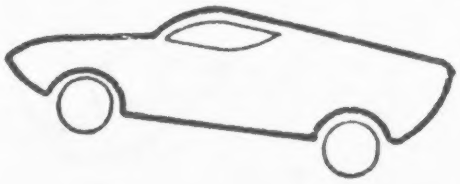
Figure A

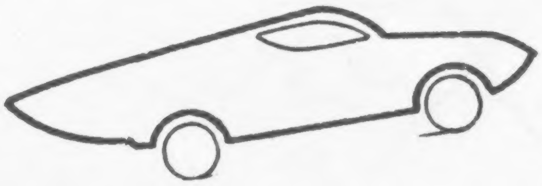












(Fold Out To Check)

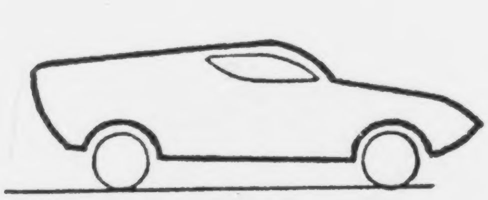
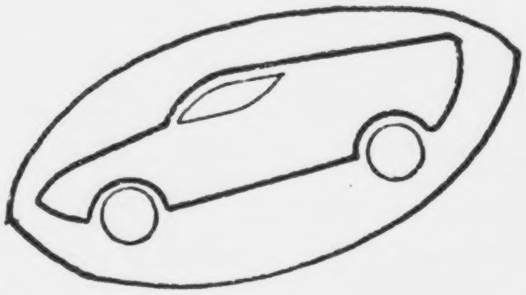
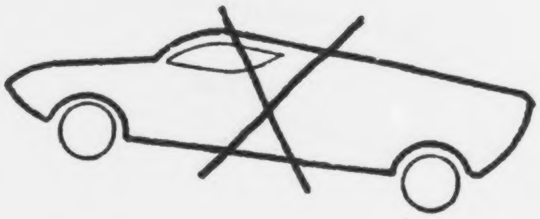
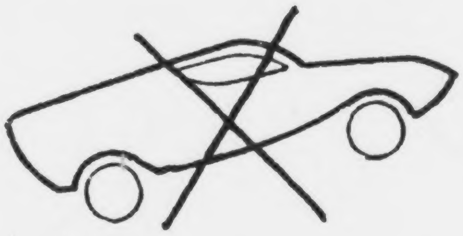
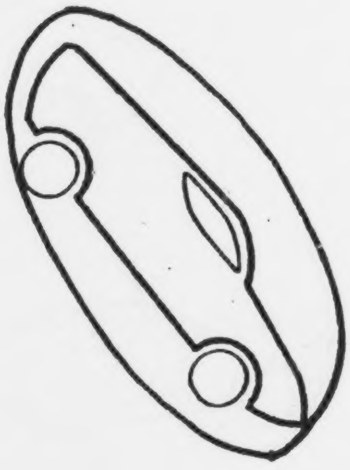


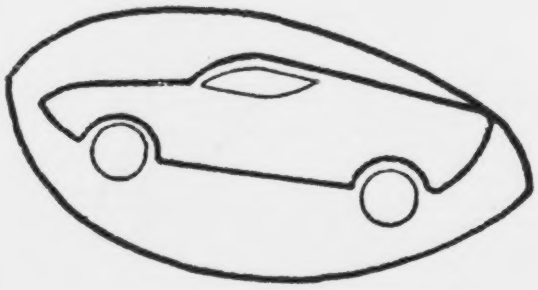
Figure A

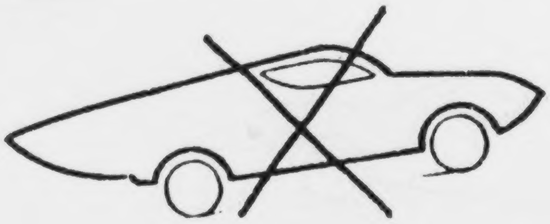




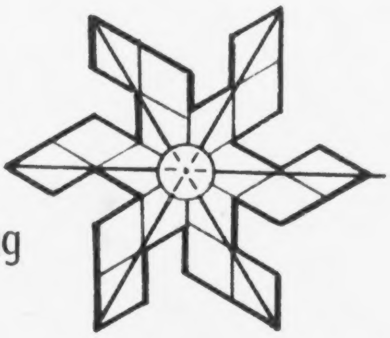




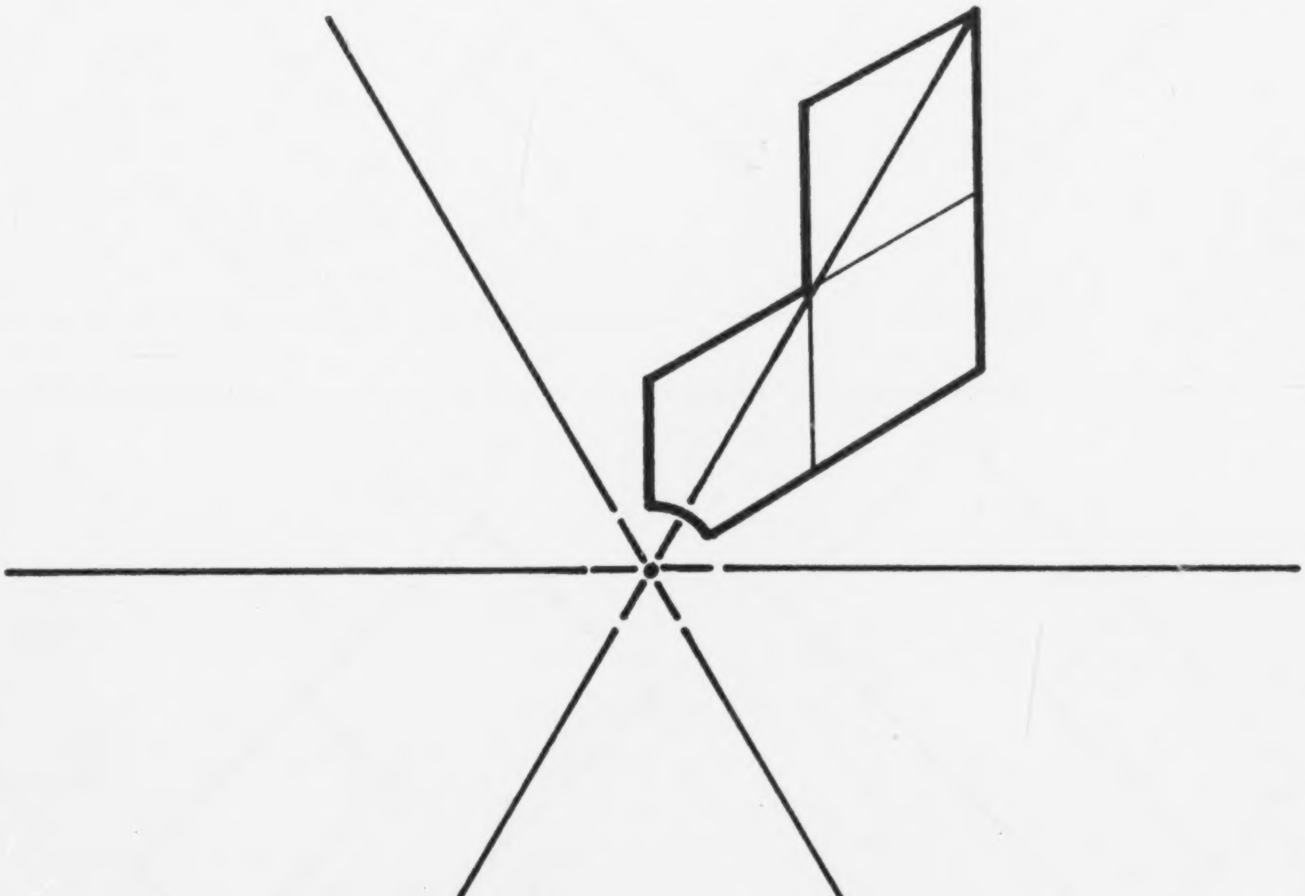




This is a snowflake.

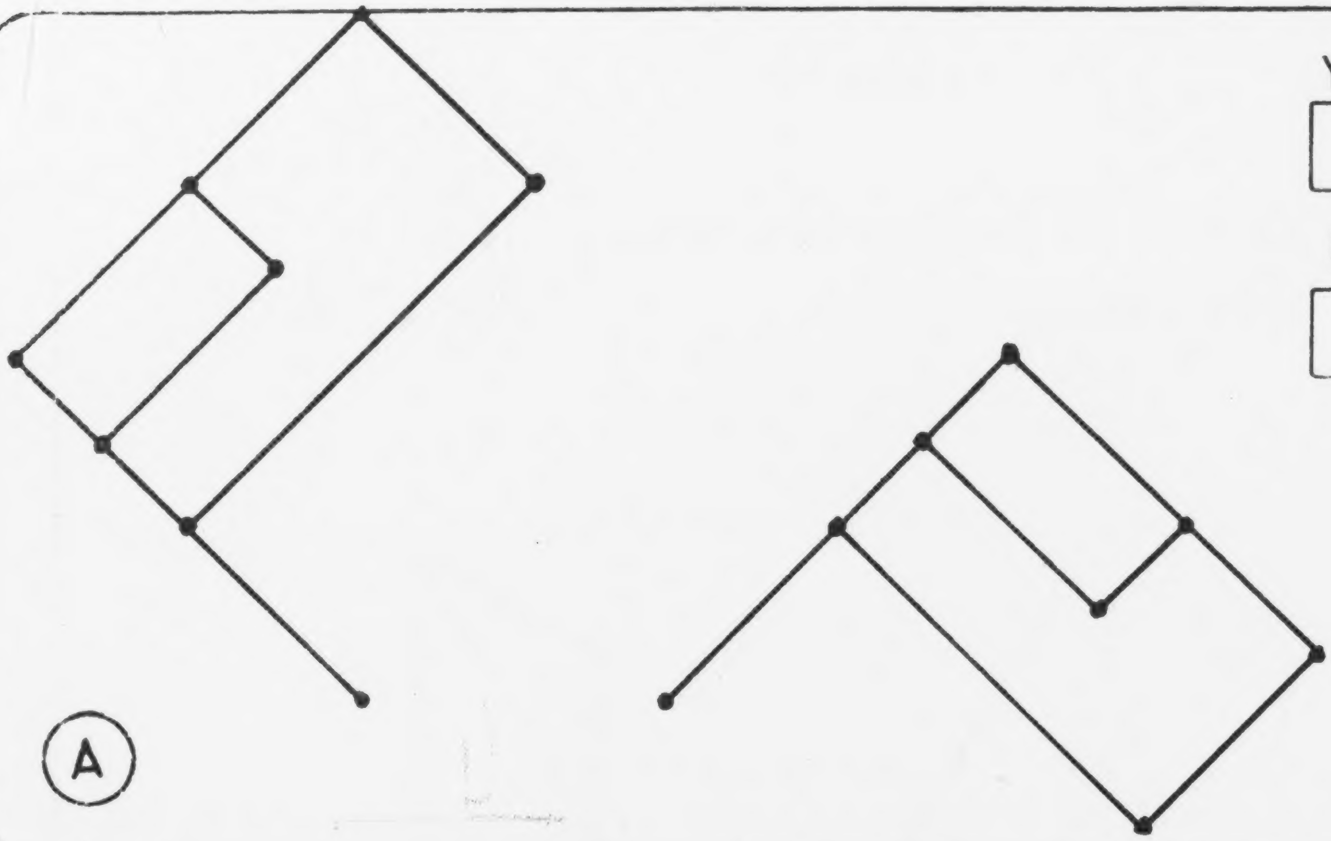


Complete the snowflake below by using tracing paper.



6

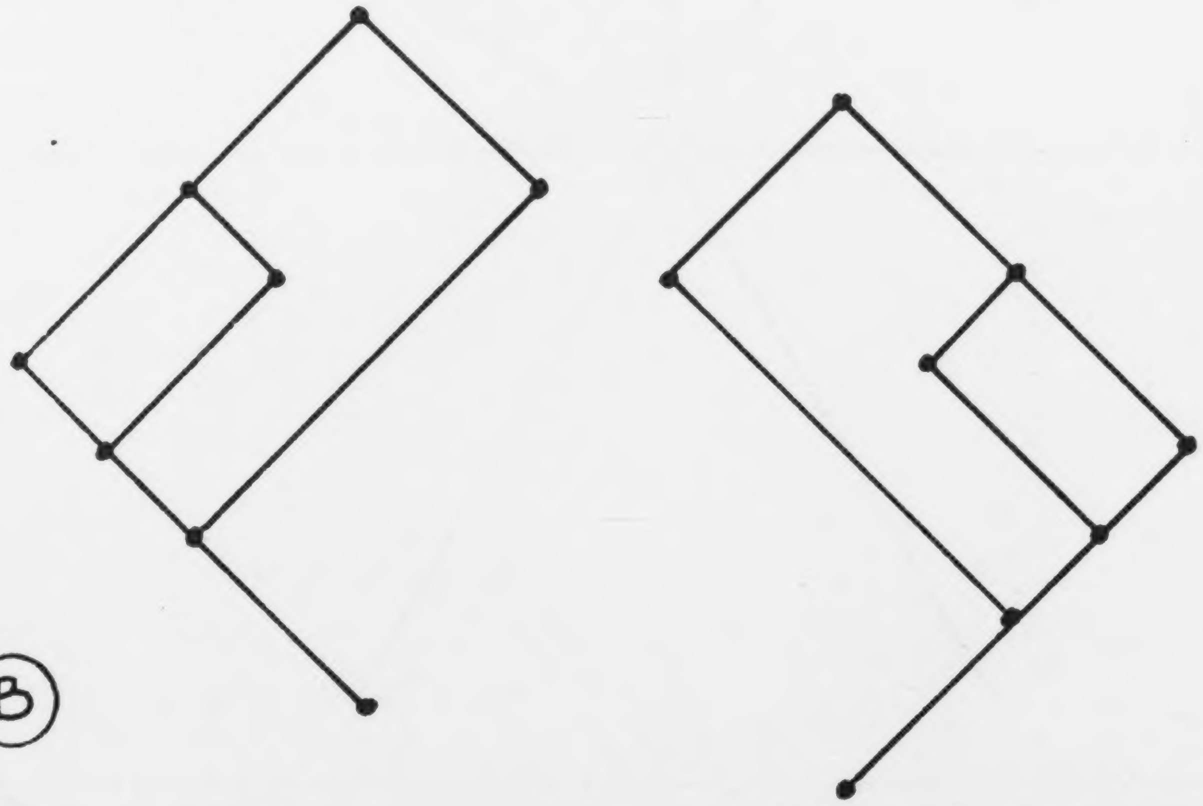
Check YES if the figures match. Check NO if they do not match. Try to do this without tracing.



YES ☐

NO ☐

Ⓐ



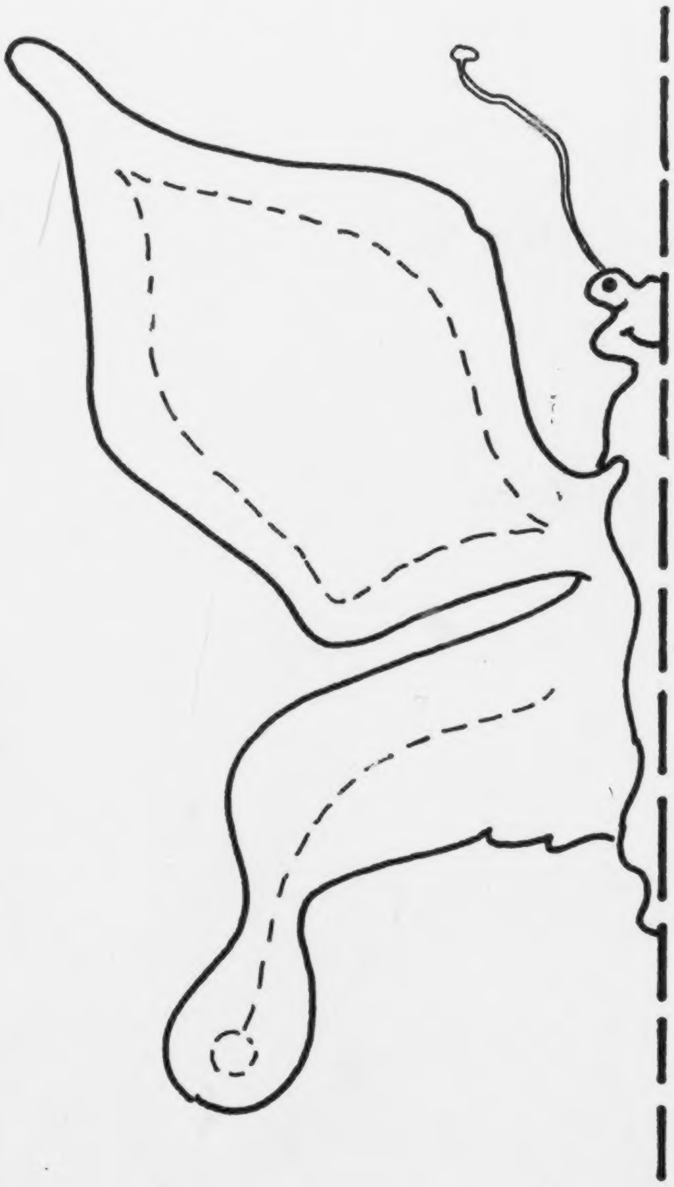
YES ☐

NO ☐

Ⓑ

Complete the picture of the butterfly.

Use tracing paper to make the other half match.



8

Circle **MG07** on your station list.

Get your next activity.

APPENDIX IV

"Toward a Master Plan of CSMP K-6 Content Development"

(April 1970 Meeting of CSMP Staff Associates)

APPENDIX IV

TOWARD A MASTER PLAN OF CSMP K-6 CONTENT DEVELOPMENT

On the occasion of the April 1970 Staff Associates Meeting this problem was presented as needing immediate attention:

> We feel that the time has come for us to get an overall picture of the mathematical content of our entire elementary program. We suggest that this task might be approached in terms of identifying unifying topics -- if there are any such -- and then showing how they blend both with one another and with other topics, which should also be identified. The overall picture should give a set of goals -- subject to later modification, if necessary -- for individual topics and for the entire program. It should also spell out the important concepts within each topic and then indicate where these overlap or complement each other. Outlines which have already been written should be taken into account to see where they fit into the total picture; decisions should be made as to whether other outlines, or position papers, should be written.

Several reasons point to the urgency of the above:

1. The program will ultimately be used in the public schools: the public school administrators have a right to know what they're buying; they must be convinced that the program covers not only what they feel is basic, but substantially more.
2. A rationale is needed both for the public and for ourselves, justifying our choice of topics and the amount of time spent on each.

3. A production schedule has been set up for the 4-6 grade materials. If such an overall plan existed we would be able to judge whether or not such a schedule is feasible and, if feasible, how much manpower is needed for us to meet its deadlines. We would also be able to judge whether and when a given phase of the schedule has been completed.
4. The secondary program needs to be planned several years in advance.
5. With a master plan to follow, our writing staff will make more efficient use of time: with better planning fewer materials will have to be discarded.
6. Staff morale is higher when one can see when a phase of work has been accomplished; this factor is especially significant in the training of new personnel.
7. There is a marked tendency to include too much in early materials unless one sees the overall plan. This pitfall was avoided in the writing of the EM materials because of the existence of such a plan.
8. Such a master plan will help the writing staff set flexible priorities for the selection of topics and the allotment of time.

APPENDIX V

DETAILED DESCRIPTION OF BOOKS 0-12 OF THE EM PROGRAM

DESCRIPTION OF EM BOOKS 0 - 12

- BOOK 0 -

INTUITIVE BACKGROUND

Book 0 plays a specific role in the EM program. It is used for those students who have not been in CSMP elementary school classes. Capable students who used the CSMP elementary school packages, designed for students of this caliber, will have had most of the experiences in Book 0 during grades 3, 4, 5, and part of 6. They should be ready to begin Book 1 (INTRODUCTORY LOGIC) sometime during their 6th year in school. Therefore, Book 0 is designed for 7th-grade students new to the CSMP curriculum who will use it simultaneously with Book 1 and Book 2 (LOGIC AND SETS) during the 7th grade and the major part of the 8th grade. Book 0 is designed to provide students with a rich intuitive background upon which the rest of the EM books can build for motivation and interpretations.

The concepts in Book 0 are presented, as much as possible, from problems and situations. Diagrams, tables, and graphs are introduced to illustrate problems and situations, and to present notions of set, ordered pairs, relations, operations, functions, etc.

This book gives practice in skills, especially arithmetic skills, that are built in connection with interesting applications, games, etc. Book 0 consists of ten chapters:

1. Operational Systems
2. The Integers
3. Sets, Subsets, and Operations with Sets
4. Ordered Pairs, Generated Sets, and Mappings
5. The Positive Rational Numbers
6. Decimals and Applications of the Positive Rational Numbers
7. The Rational Number System $(Q,+,\cdot,<)$
8. Intuitive Geometry
9. An Introduction to Number Theory
10. Probability and Statistics

At present writing, first drafts of all the chapters (except Chapter 7) and first classroom trials of Chapters 1-6 and 9-10 have been completed.

The EM writing staff feels it is undesirable for students of ages 12-13 to study purely formal matters in isolation from other parts of mathematics for the 7 months or so it may take to complete Book 1 and Book 2. To remedy this situation, the 7th-grade students beginning the EM program study Book 0 approximately three days per week and Book 1 two days per week. For most students Book 0 takes about 2 years to complete under this scheme. Book 0 above would make an excellent pre-algebra course for 7th and 8th graders.

- BOOK 1 -

INTRODUCTORY LOGIC

The material in Book 1 has been written with the following specific aims in mind:

a) To develop an accurate and efficient symbolic language suitable for expressing mathematical statements and names;

b) To develop essential notions of a proof theory;

c) To prepare the student for independent work with written material with a minimum of classroom-type instruction.

The EM materials contain a far more complete treatment of logic than can be found in current school mathematics materials. The idea of a more explicit treatment of formal logic did not, of course, originate with CSMP. Earlier pioneering projects dealt with formal logic to a degree that seemed unusual at the time. From today's viewpoint it appears that -- perhaps, with the exception of UICSM -- the logic was approached quite gingerly. It is hard to say whether these projects were responding to, or setting, a trend. In any event, even a cursory check of school mathematics texts confirms that most of them have at least one chapter on logic and sets. Indeed, the practice extends up through college-level texts and into early graduate level. It is reasonable to conclude that the writers of such higher-level texts regard this attention to logic and sets as worthwhile, and assume that their readers do not have the necessary competence in these areas, or at least need to review it.

A textbook writer has a special attitude toward language, for he knows that he will not be there to mediate his exposition when it reaches the reader. He welcomes any device that makes communication easier. No modern writer would think of giving up the precision and space economy of mathematical symbolism, and it is clear that many wish to use the precision and accuracy of logical symbolism as well. Most texts are written with the expectation that a teacher will play a role in the use of the exposition, with that role the greater, the lower the academic level of the materials. In view of the instructional procedures of the EM program, it is not surprising that we want to experiment with the use of a strong, formal language for logic and sets.

In view of the recurrent, repetitive treatment of logic from at least the 7th grade up to graduate school, it is strange that a course in logic does not have some place in the mathematics curriculum. At the college level it is still more commonly found in philosophy departments than in mathematics departments. Certainly, most present treatments of logic must be considered unsatisfactory, both as presentations of logic itself and as tools for mathematics. As one mathematician puts it, the student "must read the usual canonical chapter on logic and sets." At the school mathematics level he must read it again and again for, although the student matures, the successive canonical chapters do not show a like development. They deal with the same dreary truth tables, inadequate semantic discussion of proof, with sometimes a bit of predicate language or syllogistic. The set theory treats the same dreary Venn diagrams.

There is nothing inherently dreary about truth tables or Venn diagrams the first time through. They become dreary with repetition and when the student discovers that, generally, he can leave the subject in the first chapter, since the rest of the book makes no essential use of it or is even inconsistent with it. In EM we want to try the experiment of teaching logic to students with the expectation that they will learn it and use it immediately and continually as they do the other things they learn.

A second role of logic is to present some sort of proof theory. For years we have taught students about proof in a way that is analogous to the so-called "direct method" for teaching languages, i.e., by exposing them to lots of proofs and requiring them to construct replicas. In the geometry class there is often some analysis of what a proof is, but the analysis and description of proof is invariably inconsistent with the actual proofs that appear in the text. If the student is to go beyond replicating models, he must abstract for himself some concept of proof from the examples he sees, from inadequate descriptions, and always subject to the authority of the teacher. This is hard on the student, since teachers do not at all agree on what a satisfactory proof form is. Of course, many students do eventually attain a workable concept of mathematical proof, but we believe the length of time spent to reach this goal is far too long, and the number of students succeeding far too small.

To be consistent with our aim to develop the student's resources to check the correctness of his own work, we must provide him with some criteria for judging his proofs other than approval by a teacher. The plan is to break the large sequence of inferences down into very small immediate inference rules, leaning very heavily on the student's command of language for acceptance of the reasonableness of the basic rules. Since at this stage we are necessarily concerned with language and form, the mathematical content of Book 1 is low compared with that of later books. Also, we do not want to complicate the problem of learning what a proof is with the much harder problem of learning how to find proofs. Thus, most of the proofs the student works on at the beginning are relatively easy ones. The initial proofs that the student sees and writes are quite long but, as the student gains experience, immediate inferences are grouped in larger chunks, are abbreviated, or left tacit. It is the aim that, eventually, the overt manifestations of the logical machinery will wither away from the student's written proofs, leaving the stage to the mathematical content. The students learn several colloquial styles of proof, and eventually write proofs that look like those any well-trained mathematics student would write who liked to use logical symbolism.

We are well aware that the proofs at the levels of Books 1-3 bear little resemblance to the colloquial abbreviated proofs of mathematicians. The difference is not much greater, however, than that between beginning algebra and higher mathematics. The proofs of high-level mathematics are addressed to the profession, and it is futile to try to check them step by step in the way we do for students at age 13. It is still the case at any level that not every string of theorems and inferences is a proof: that proof, belief, and truth are not identical; that one needs to know what the presuppositions are. One cannot use the mathematicians' "it is clear that" with students who cannot see how clear it is, who confuse "$P \Rightarrow Q$" with "$Q \Rightarrow P$", or make mistakes in immediate inference. It seems evident that for the majority, before they can understand sophisticated proofs, they must be able to understand simple ones, and one way to understand simple proofs is to analyze them. This we do.

Book 1 is largely concerned with developing a statement calculus. Some of the language of predicate calculus is introduced (but not its proof theory), such as open sentences, open names, and quantifiers. The notion of an axiomatic system and model is introduced.

Our proof theory of the statement calculus is a form of natural deduction and, in effect, allows all tautologies as axioms. We do not, at this stage, want to have the student engage in the kind of proofs necessary to develop the system from a small set of inference schemata. Tautologies are identified by the semantic device of truth tables. To take the tautologies as axioms makes the proof theory essentially trivial, and the students are aware of this in time, but find it to have practical value. They are promised that the same proof theory will soon be extended into areas where it is essential.

The time scale for completing Book 1 is 14-18 weeks for most students.[1]

By this time the students are expected to have made substantial progress along the lines set out at the beginning of this section. They are expected to be able to decide, for any statement formula that might reasonably arise, whether or not it is a tautology. They are expected to be able to express the forms of reasonable English sentences in symbols to the extent that statement calculus language can do this, and to be able to translate from symbols to English. They are expected to be able to construct abbreviated proofs, and to be able to put in the tacit steps if asked. (They are not required to write out any complete unabbreviated proofs at this stage.) They are expected to be able to handle indirect proof and proof by cases, and to discharge assumptions properly by the deduction theorem or by indirect inference. They are expected to understand what a variable is, the processes of substitution and replacement, what it means to say that one statement is in the form of another, or that two statements have the same form. They are expected to replace and substitute accurately, properly, and appropriately, and to recognize same form in reasonably complex formulas. These latter skills are basic in elementary algebra, and indeed in all of mathematics, and the curriculum time spent here pays dividends outside of the logic, such as in Books 3 and 4.

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[1] This is the time estimated for completion when not taken simultaneously with Book 0. The average time needed for Book 1 when studied only two days per week is 8-10 months.

Book 1 has been revised several times, and has had some field testing. It is now in usable book form with appropriate index and appendices. Further revision will be done, but it is low on the list of priorities in the total writing effort.

- BOOK 2 -

LOGIC AND SETS

Book 2 continues development of the themes cited at the beginning of the section on Book 1. The symbolic language is extended to the language of sets. Proof theory is extended to include basic quantificational inferences that concern the universal quantifier. Systematic abbreviation of proofs continues.

In the EM materials hardly any major topic is fully presented in just one book. The basic logic started in Book 1 is not essentially completed until Book 4. The logic is also extended somewhat in Book 6 and again in Books 7-8 when they deal with induction and recursion. Similarly, the treatment of sets is not confined to Book 2.

Development of predicate language continues in this book, but is applied increasingly only to mathematical material. Necessarily in Book 1 there was heavy emphasis on natural English but, even there, the students were not given very complex English sentences to translate. The usual four categorical forms[1] were thoroughly treated, but nothing much more complicated. The reason for this is that the logic is being developed for use with mathematics and there is no reason to tackle the problem of translation into predicate language of complex English sentences that do not concern mathematics. Mathematical language does involve multiple quantifications of multiple pronouns (i.e., variables), but the informal mathematical language is generally less ambiguous than natural English, and closer in form to the symbolic form.

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[1] All M's are P's; some M's are P's; all M's are not P's; some M's are not P's.

Proof theory is extended to include inference schemes for universal instantiation and generalization. Existential Instantiation is treated systematically in Book 3, and existential generalization in Book 4. It would be neater, perhaps, to present all the four quantificational inference schemes together, but there is no occasion for continued use of the existential schema in the material of this book. The process of further abbreviation in proofs of the statement calculus machinery continues.

It is assumed that the students have gained much experience with informal treatment of sets in Book 0 and in earlier studies, and that they are familiar with various collections of numbers and other mathematical objects that we wish to use to interpret the set theory. Thus, Book 2 is concerned primarily with pulling the informal notions together in a unified system. A strong substitution axiom is assumed: $\left[x = y \wedge F(x)\right] \Rightarrow F(y)$. Equality is interpreted in the usual way. An even stronger comprehension axiom is assumed: $x \in \{y \mid F(y)\} \Leftrightarrow F(x)$. We are well aware that the assumptions we make are too strong for a consistent theory, but do not feel anything is gained by introducing 13-year olds to set paradoxes. For a long time to come the sets they will work with are well-behaved. The system adopted does not commit the student to any particular ultimate set theory, but can probably be replaced most readily by the system of Quine's "New Foundations". A strong metatheorem on the substitutivity of the biconditional is assumed without proof. The time for completing Book 2 is 10-14 weeks for most students, when pursued five days per week. When studied only two days a week, as is the case until Book 0 is finished, 4-5 months are usually required.

At the completion of Book 2 it is intended that the students

a) can read and write set language easily and accurately, and can understand it without first translating into natural English;

b) can prove theorems of the type that occur in the book, can use abbreviated proof patterns for dealing with proofs that consist of a sequence of equivalences or equalities, and have learned how to discuss proofs with one another and with teachers;

c) have deepened their understanding of axiomatic organization;

d) have an understanding and reasonable facility with the usual Boolean operations involving intersection, union, complementation, universal set, empty sets, set difference, the subset relation.

Finally, it is expected that the students have adjusted to, and accepted, the responsibility for using the materials independently of the teacher. It should be remarked that students who started Book 1 together will almost certainly complete Book 2 at different times. It is expected that they will have adjusted to this situation as well.

Book 2 exists in usable form. Further revision is contemplated but has low priority in the writing schedule.

- BOOK 3 -

INTRODUCTION TO FIELD THEORY

Book 3 is designed

a) to treat in detail development of an axiomatic system as a means of organizing a body of mathematics, much of which the student is already familiar with;

b) to treat interpretation of an abstract system:

c) to provide the basic concepts of algebra;

d) to help the student master the skills normally learned in a first course in algebra;

e) to extend proof theory to include existential instantiation.

The vehicle is the axiomatic theory of a field. A field is presented here as a set and two primitives, addition and multiplication, subject to the usual axioms. Clearly, the structure of a group is simpler than that of a field; however, the students have been working with two-fold operational systems for some time, and it is more natural in terms of their background to work with fields (or rings) than it is with one-fold operational systems, such as groups. Furthermore, we do not want to delay developing mastery of normal algebraic skills, since that would also unduly delay applications of mathematics. Some notions of group theory are introduced in exercises, but a fuller treatment will be delayed until groups are needed in some essential way.

The theory is interpreted in terms of the finite modular fields, the rational numbers, and the real numbers, although the student's concept of real number is vague at this stage. The algebraic skills the students develop are not conceived by them as just things to do with rational or real numbers, as is the case in traditional algebra; for example, they do extensive computational work in the finite modular fields as well. Since they see and work with more extensions of the abstractions commutativity, distributivity, etc., than they would in a traditional algebra course, we expect them to have a more mature understanding of these abstractions.

The usual skills with integral exponents are mastered. The operation expressed by "nx", where n is an integer and x a field element, is systematically developed as iterated addition. Integral exponents are similarly developed in terms of iterated multiplication. Both are described recursively, but all proofs are informal, since we have chosen to delay treatment of mathematical induction until quantification and notions of function are well in hand. Systems of equations in more than one variable are delayed until ordered pairs are treated in Book 6.

Development of the symbolic language includes precise treatment of the notion of free and bound occurrences of a variable in a formula. Proof theory is extended to include existential instantiation, which entails some adjustment in the inference scheme for universal generalization.

On completion of Book 3 it is expected that the students

a) have an accurate notion of what an axiomatic system is, and what it means to interpret it;
b) have taken the first steps in learning what it means to "mathematize", that is, to set up a mathematical model of something;
c) can factor, multiply linear forms, solve linear and quadratic equations, and do other familiar manipulations of elementary algebra, but understand these as manipulations applicable in any field;
d) understand the two different uses of variables

in mathematics, exemplified by free and bound occurrences;

e) can prove theorems comparable to the ones they have been studying, involving the first three quantificational inference schemes, in reasonably abbreviated style;

f) are virtually independent of a teacher for checking the results of routine algebraic computations.

Book 3 takes about 5-6 months to complete. It is in book form in a preliminary edition, properly indexed. Revisions of Book 2 will necessitate some adjustments in Book 3, but they will be minor.

- BOOK 4 -

ORDER IN FIELDS

Book 4 is a natural extension of Book 3 and continues to develop the aims stated in the previous section. The book completes the introduction of most of the mathematical notions normally encountered in elementary algebra, and completes introduction of the basic pure predicate calculus. Except for mathematical induction and recursive definition, little more will be added to the logic for a long time in the series.

The main purpose of the book is to introduce order into field structure. We see no good reason why students should not learn about equations and inequalities at the same time. The average college freshman today is reasonably confident about handling equations, but has some diffidence about handling inequalities, especially those involving absolute value. Some students have come to conceive a number as having three states -- its plus state, its minus state, and its "numerical value", that is, the number without its "sign". Such notions, and the diffidence about inequalities, are a hindrance to learning analytic geometry and calculus.

Order is axiomatized in a familiar way in terms of an assumed subset of positive elements closed under the primitive operations, etc. Order is not defined as a relation, that is, as a set of ordered pairs. This matter is considered in Book 6. The usual theorems about order are identified and proved. More proofs are left to the student than in the earlier books. With the machinery available, it is not difficult to handle the notion of equivalent equations, which is often a problem for students in traditional elementary algebra. In this book the idea of equivalent inequalities also causes no problems. There is full treatment of absolute value, including proof of the triangle inequality. Square roots and quadratic equations in ordered fields are fully treated. The problem of attaching meaning to "$\sqrt{a}$" in a finite field and the impossibility of such a field being an ordered field are dealt with.

The proof theory is extended to include existential generalization. This was delayed until this point because, even though there were a few occasions where it was needed, there was not occasion for its continued use until now. All the quantificational inference schemes are reviewed and the possibility of confusion of bound variables discussed.

The kinds of idealizations and assumptions that must be made to apply algebra are dealt with in detail, and problem sets include many exercises dealing with time-rate-distance, mixture, compound interest and annuities, and other less traditional problems.

At the completion of this book we expect the students

a) to know what a field and an ordered field are, and to be able to carry out for any field the processes normally associated with elementary algebra;

b) to have a firmer grasp of the notions of axiomatic system, interpretation and model, (they will not as yet have well developed ideas about properties of axiomatic systems, such as independence, consistency, etc.);

c) to have made a start on developing competence in handling inequalities comparable to that for equations;

d) to be able to apply algebra to simple problems;

e) to have some picture of logic as a system, and, in a theory, to be able to prove appropriate theorems in a good abbreviated style that suppresses the more familiar logical machinery and presents the mathematical ideas with proper prominence.

Book 4 takes the students about 3-4 months to complete. It is in book form and currently under revision to include problem sets and to lighten what is now too much a Satz-Beweis style.

* * *

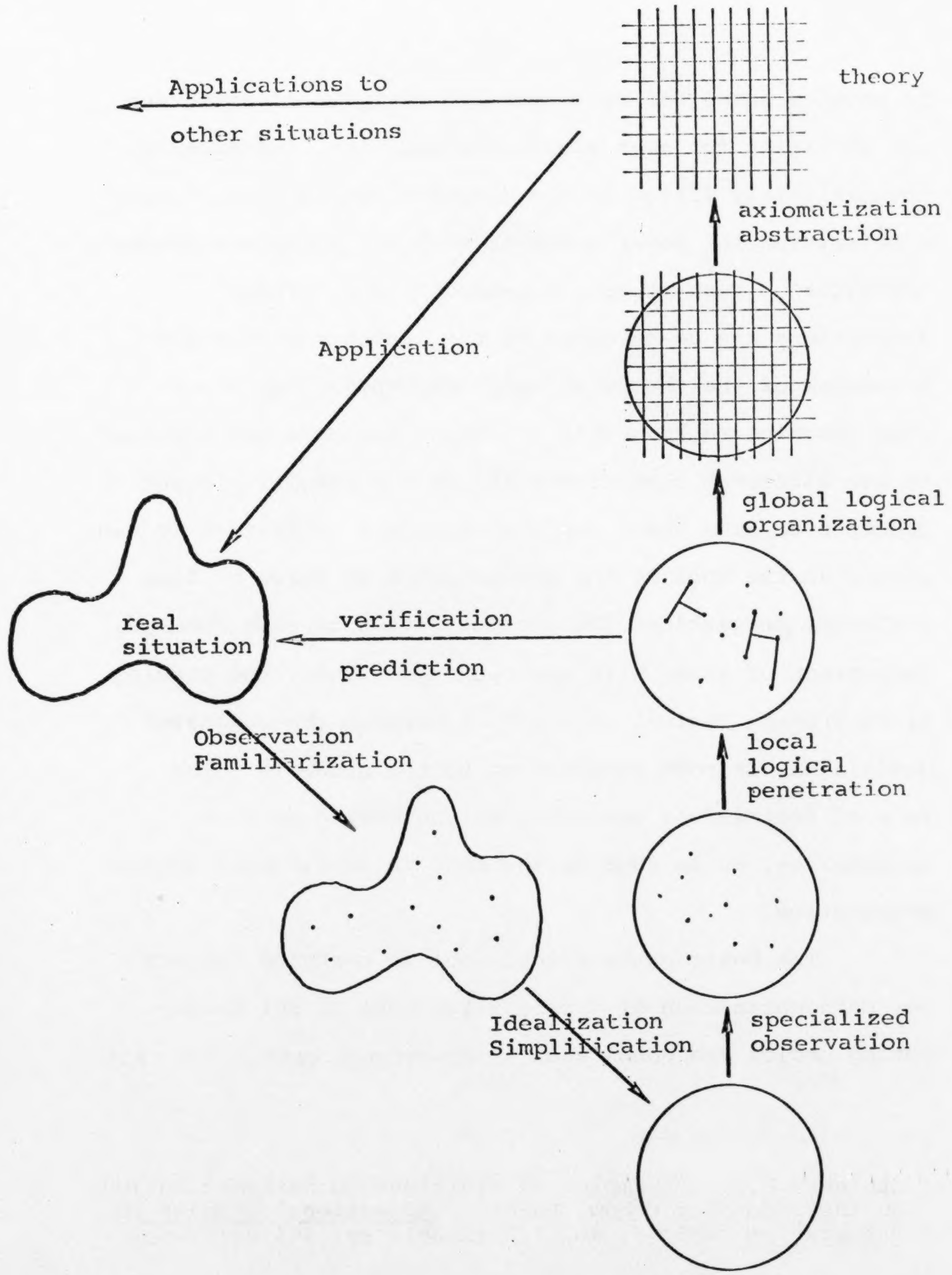
- BOOK 5 -

AN INTRODUCTION TO MATHEMATIZATION: A THEORY OF VOTING BODIES

The purpose of this book is to let the students actually participate in a mathematization process which leads from situations outside of mathematics to a mathematical theory. This process, which is a special type of mathematical model building, plays an important role in modern applications of mathematics, particularly in fields other than physics, such as economics, the social sciences, biology, cosmology, etc.

The situations to be mathematized, which we have selected for Book 5, occur in human society when a group of people must make decisions by voting. It is possible to develop a small and elementary, yet non-trivial, theory of voting bodies in such a way that many significant steps of a mathematization process can be properly exemplified. The steps which play a particular role in the development of the theory of voting bodies can be found in the diagram on p. 23, which gives a simplified and rough picture of the entire process.

The students are kept engaged in actually performing these and other steps by interspersed exercises,


Applications to
other situations
theory
axiomatization
abstraction
Application
global logical
organization
real
situation
verification
prediction
local
logical
penetration
Observation
Familiarization
Idealization
Simplification
specialized
observation

by problem sets, and by a detailed analysis of different possibilities for each step. One activity, for example, not explicitly listed in the diagram, consists in finding a definition for power concepts such as "powerless member", "dictator", "veto-power", "chairman", etc. These definitions are to be given by the students within the framework of the theory already developed. For classroom teaching we know that different students are inclined to use different characteristics of the intuitive power concepts to make their own definitions.[1] This fact is simulated in the book by the presentation of three or four different suggestions for the definition of each concept. Comparison of these with his own suggestions, and proving or disproving logical equivalence between the suggested definitions is most challenging to the students. The role of definitions ceases to be the boring part of mathematics, as is usually the case in traditional school mathematics.

The basic mathematical tool in carrying through our mathematization of voting situations is set theory. Voting bodies are considered as structured sets. The fact that a set of voters -- as the Security Council of the United Nations -- can get a voting structure by only determining certain subsets to be winning coalitions, without talking about vote distribution and a majority rule, leads to two voting-body concepts. On an elementary level this is a striking analogy to the difference between topological and metric spaces in advanced mathematics. The methodological meaning of this difference can be explained fully to the students by the unexpected result that there are Security-Council-type voting bodies for which no such vote distribution and majority rule exist which determine the given winning coalitions.

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[1] Steiner, H.G., "Examples of Exercises in Mathematization on the Secondary School Level". Educational Studies in Mathematics, Vol. I, No. 1/2 (1968), pp. 181-201.

After the students have been involved in the development and full construction of the theory, they are asked to rewrite the whole body of results in a strict axiomatic-deductive way; appropriate examples are familiar to the students from the preceeding books. Here they may, and are urged to, re-arrange the order of definitions and theorems according to economical or aesthetic viewpoints. This is another important activity of mathematicians to which students are hardly ever exposed. In the final theoretical product as a sequence of axioms, definitions, theorems and proofs, no reference is usually made to the motivations which come from the analysis of the original situations and of the related intuitive concepts.

In this way the students learn also, through personal experience, the difference between the creative development of a theory and its presentation as a finished piece of mathematics.

In the last part of Book 5 another important feature of mathematical model-building is exhibited. One discovers cases which the theory surprisingly does not cover. The problem then arises of finding an appropriate extension of the theory.

At the completion of Book 5, it is expected that the students

a) have a notion of some characteristic steps of a mathematization process; understand especially that axiomatization can be considered as a creative act of concept formation;

b) understand the possible role of a definition as the formulation of an intuitive concept within a given framework; understand the problem of adequacy of a definition and the distinction between the "explicans" and the "explicandum";

c) understand the difference in style and meaning between developing a theory and presenting it as a finished product.

Book 5 takes the students about 2-3 months. Its first draft has undergone some changes in sequence and a little more extension of the opening chapter.

- BOOK 6 -

RELATIONS

The purpose of this book is to develop an understanding of relations, particularly as a basis for the next three books. The threads of earlier aims continue here.

Relations are treated as a further development in set theory. Ordered pairs are defined and set-builder notation extended to accommodate them by defining "$\{a \mid C\}$", where "a" is a term (open name) and "C" is a condition (open sentence). Relations are defined as sets of ordered pairs, and usual notions of domain, range, field, converse, composition, image of a set, etc., are defined and developed. Properties, such as transitivity, symmetry, antisymmetry, etc., are treated. Particular attention is devoted to notions of partial order, total order, and ordered set. The notions of "$<$" and "$\leq$", introduced in Book 4, are reviewed and reconciled with the notion of relation as dealt with in the present book.

Particular care is taken in presenting equivalence relations, equivalence classes, and partitions. These, of course, are important matters in any event, and are vital preparation for Book 8. Book 6 closes with extensions of set union and intersection to manifold unions and intersections, which will be needed in Book 9. There is some preliminary introduction to the notion of closure of a set with respect to a relation. The style of presentation is more colloquial than that of any earlier book. The proofs are more abbreviated and, except for the language, the logical machinery is less overtly evident. Now we dare to use such phrases as "it is clear that" or "the rest is routine", and to leave some proofs unfinished. The student understands that he is responsible for the proofs, and should actually carry out any unfinished proofs to the point that he is sure he can finish the job.

There is much use of various modes of picturing relations - by arrow diagrams, graphs, matrices or tables, and other devices. The utility and appropriateness of such devices are discussed and the student is encouraged to invent his own.

At the completion of this book the student is expected to have made a substantial start on the important ideas of relation, order, equivalence relation, and to be able to read and use the various ways such matters are expressed. The notion of relation is a unifying concept, as is function, and we expect the student to view some of his earlier work in a more comprehensive way from this viewpoint. He is expected to have matured in the matter of proof-style and in the matter of finding proofs. He is expected to have matured in taking responsibility for his own learning.

Book 6 is estimated to take 4-5 months for most students to complete. It is now in book form. Some revision of problem sets is desirable but further revision is of low priority.

- BOOK 7 -

FUNCTIONS

This book builds naturally on the material of Book 6. A function is defined as a special kind of relation, and concepts such as domain, range, etc., already treated in Book 6, are reviewed in the new setting. The common ways of looking at functions are dealt with, as well as the various standard notations and ways of talking about functions. Every effort is made to use the notation and concept of function that are appropriate to the job at hand.

Informal uses of functions in earlier work are revisited. Statement calculus connectives are reviewed as functions from $Z_2 \times Z_2 \rightarrow Z_2$ and results are applied to switchings and relay circuits. Field operations of addition and multiplication -- which were introduced in Book 3 as primitives -- are recast as functions or operations.

Operational systems, and more general systems with structure, are treated in a preliminary way, with notions of group and semi-group used as examples. No deeper treatment of the group concept is attempted in this book. The concept of structure-preserving map (e.g., homomorphisms, order preserving) is introduced.

There is an extended section on sequences, including the usual arithmetic and geometric sequences and series, recursively defined, with commercial and other applications.

Finite sequences are used to develop various counting models, including much of the material referred to in college algebra books under the tag "Permutations and Combinations". Matrices are defined as finite double sequences, and the basic matrix operations treated. They are applied to solutions of systems of equations, but no general theory is attempted at this stage; rather, matrices are treated as useful in dealing with certain types of problems.

Mathematical induction is indispensable in dealing with sequences, and is introduced in preliminary form in this book. Its use in proofs is limited to cases where the induction hypothesis is not quantified. Only weak induction is used at this stage. Repeatedly, in earlier books, proofs were postponed until the time when mathematical induction was treated. This earlier work is revisited and key results dealt with rigorously.

The Σ-notation is defined recursively and the basic properties proved by induction. The relation of an arithmetic sequence to its sum-sequence (or series) is generalized in a section on Difference Sequences, with many applications requiring use of matrices, binomial coefficients, etc.

The material outlined above is currently being put in preliminary book form. That material will be completed with more applied problems and with the last section, now in rough draft, on various standard functions and their applications.

On completion of this book the student is expected to be at home with the various languages and notations used to handle functions, and the various different ways of using functions that are needed in the books immediately to follow. It is expected that he will have accepted the notion of a function as a mathematical object and that sets of functions may be fruitful objects of investigation.

It is expected that the student will have made substantial progress in the matters of problem solving and applications of mathematics; that he will have increased his repertoire of techniques.

Finally, it is expected that the student will have a sound preliminary grasp of recursive definition and of mathematical induction as a proof form.

- BOOK 8 -

NUMBER SYSTEMS

In this book, the natural numbers will be modeled within set theory. Addition of sets will be defined, 0 defined as $\{\emptyset\}$, 1 as the class of all singleton sets, and the whole numbers as the closure under addition with 1. With one additional axiom, the usual properties of whole numbers can be developed, as well as the basic theorem on induction.

Induction as a proof-pattern will be treated fully, including the matter of quantified induction, hypotheses, and strong induction. Theorems accepted earlier without proof will be revisited (e.g., general associative and commutative laws).

The usual extensions to integers and rational numbers will be carried out. This is usually a tedious matter, but we do not anticipate that it will be in this case, with the background the students will bring to the subject. The ordered field of real numbers will be developed via Dedekind cuts. The notion of least upper bound will be explored, as well as preliminary notions of limit point, and limit of a sequence.

Preliminary notions of cardinal similarity and the cardinal number of a set will be introduced in this book. Systems of numeration with various bases will be dealt with.

There will be much work with decimal approximation to real numbers and with iterative processes leading to approximating sequences.

Book 8 will be one of the shorter books -- substantial parts of it exist in rough draft manuscript at present.

- BOOK 9 -

FINITE PROBABILITY SPACES

This book is finished in a preliminary edition. Two classes were taught part of this material during the 1968-69 school year and the book was produced in connection with this teaching. The students are familiar with the mathematics which will form the content of the first eight books. This procedure can utilize student reactions from the very beginning. The book is devoted to probability theory restricted to finite outcome sets.

There are many reasons for including a book on probability at this stage of the EM series. Probability theory is an important field of mathematics with numerous applications. At present much research work is done in this area, so this book will give the students contact with a field of mathematics that is presently expanding. This book also gives the student the opportunity to realize the power of his mathematical knowledge and skills developed so far. Furthermore, Book 9 will give the students the possibility to apply their knowledge of logic, sets, relations, and functions. As most of the concepts in probability theory are motivated by real-world situations, this book will also give the students concrete experiences on the interplay between the real world and mathematical model building (mathematization). Applications from various fields as statistics, operations, research, science, technology, and so on, will be included.

After an introductory chapter dealing with some history of probability theory, and some practical activities serving as empirical background, the definition is not restricted to a uniform probability measure, which is often the case at the high school level. However, once the basic theorems are proved, this case is studied carefully, and some basic combinatorial results are derived. Conditional probabilities are treated and used to motivate the definition of the independence relations.

Usually, real-valued mappings of probability spaces under the name of "random variables" are given a central role in the development of probability theory; here, general mappings are studied first, and later, after real probability spaces have been introduced, real-valued mappings are studied at length. For the general case, marginal and independent probability spaces are defined with the aid of projection functions.

Concepts especially related to real probability spaces, like expectation and variance, are defined. The "algebra of expectations" is also developed. Independent functions are defined. Special probability distributions -- like the binomial and hypergeometric distributions -- are studied, and various applications of these distributions are presented.

The last chapter contains a comprehensive review of the entire book, followed by a set of review problems. Problems of different degrees of difficulty are presented in two distinct sections. This will strengthen the students' problem-solving ability and will also expose them to important applications from operations research, reliability theory, search theory, etc. Finally, the book discusses the extension of the theory to infinite countable outcome sets and to outcome sets which are intervals in $\mathbb{R}$ or $\mathbb{R} \times \mathbb{R}$ (geometrical probability).

In preparation is also some supplementary material dealing with variance, Chebyshev inequality, conditional expectations, and finite Markov chains.

An important problem is the real-world interpretation of probability concepts. Here, this is done mostly in terms of the "relative frequency-limit" interpretations. Other interpretations, however, are also mentioned.

- BOOK 10 -

ELEMENTS OF GEOMETRY

This book is in the writing stage. Exploratory teaching in the 1968-69 school year was devoted to studying students' abilities in grasping the basic ideas, working exercises, and solving problems.

The mathematical and methodological education which the students have had before Book 10 puts us in a very favorable position with respect to the treatment of geometry. The students are already experienced in distinguishing between, and arguing about, real and ideal situations, mathematical theories as formal systems, and interpretations or applications of theories inside or outside mathematics. This allows us from the beginning to deal explicitly with the full spectrum of geometrical considerations, from the intuitive and empirical to the most abstract formal representations. One of the greatest difficulties in traditional geometry teaching consists in the transition from a purely intuitive (and hence incomplete) understanding of geometrical concepts and statements, to the mathematical theory and its axiomatic anchoring. Students in the EM program should be able to comprehend from the outset in Book 10 what it means to begin with a phenomenological analysis of the intuitive and empirical space and get to one or more theories about certain aspects of this space. They should also be prepared to understand different methodological viewpoints in the history of geometry.

Book 10 will give an introduction to three-dimensional Euclidean geometry with emphasis on the affine structure. It will have an introductory chapter with some history of geometry. It will then proceed to a mathematization of basic phenomena in the intuitive space and plane geometry. At first this will concentrate on concepts like point, line, plane, and the incidence relations. Much emphasis is laid on the cultivation of spacial intuition by the students: this is done by discussions and by problems which need pre-analysis by drawings, interpretation, and relation to physical realizations. At the beginning the attitude is such that points, line, and planes are considered as three independent kinds of objects. This so-called "three-sorted" approach allows a greater number of more natural interpretations of the incidence axioms, as well as a natural way to discuss the projective extension of affine planes and spaces.

The mathematization of incidence phenomena in the plane and in space gives many opportunities for exercises in translating statements from natural English into formal language, and vice versa, and in analyzing the meaning of concepts depending on the state of the axiomatization process.

The axiomatic analysis of the order concept will include the important notion of orientation in the plane and in space.

Congruence will not be treated in Book 10 but is left for Book 12. The intent of Book 10 is to present a very careful axiomatic development of a fragment of geometry, and to insure that the students have a good historical perspective. Brief concluding chapters on projective and hyperbolic geometries are included.

In connection with the investigation of models, especially of finite models for two- and three-dimensional incidence and affine geometry, some fundamental concepts of the axiomatic method, especially semantical ones, will be explained or exemplified, such as consistency, independence, completeness, and the soundness and completeness of the underlying logic.

- BOOK 11[1] -

GROUPS AND RINGS

1. Elements of Group Theory

   i) Definition of semigroup and group.
   Examples: $(Z/(m),+)$, $(Z,+)$, $(Q^{*},\cdot)$, $(Q^{+},\cdot)$, $(Q-\{1\},\cdot)$ where $x \cdot y = x + y - xy$.

   ii) Uniqueness in groups of identity element, idempotents, inverses. Generalized associative law. Exponentiation as tie with Z.

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[1] This is the preliminary outline of Book 11 which is now in manuscript form.

iii) Fermat's Little Theorem as "germ result":

Compute 3^0, 3^2, 3^3, 3^4, 3^5, 3^6 (mod 7)

Repeat, a longer but more fruitful way, emphasizing use of cancellation:

$$\begin{array}{lcccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 3\cdot 1 & 3\cdot 2 & 3\cdot 3 & 3\cdot 4 & 3\cdot 5 & 3\cdot 6 \\ & \rotatebox{90}{$\equiv$} & \rotatebox{90}{$\equiv$} & \rotatebox{90}{$\equiv$} & \rotatebox{90}{$\equiv$} & \rotatebox{90}{$\equiv$} & \rotatebox{90}{$\equiv$} \\ \text{(permutation)} & 3 & 6 & 2 & 5 & 1 & 4 \end{array}$$

So, $1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 = 3\cdot 1\cdot 3\cdot 2\cdot 3\cdot 3 \;.\;.\;.$

$$1 \equiv 3^6 \pmod{7}$$

Then we get for any prime p, (Fermat's Little Theorem) $a^p \equiv a \pmod{p}$ for every integer a. This work leads "naturally" to:

a) Euler-Fermat Theorem, composite modulus.

b) Cayley's Theorem, permutation groups, permutation representation and isomorphism of groups.

c) Cancellation semigroups.

d) Cyclic groups.

e) Exponent set and period of an element.

(These last two items provide natural motivation for Lagrange's theorem on orders.)

iv) Subgroup. Isomorphism. Injection.

Examples: $(Q \bmod 1, +_1)$

$$(\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\}, +_1) \cong (Z/(n), +_n)$$

Period of element. Period of group versus order of group. Cyclic groups and subgroups.

v) Permutation groups. S_n. Cayley's Theorem.

vi) Cosets and Lagrange's Theorem. Index.

vii) "Division" by subgroups. Normal subgroups. Factor groups. Homomorphisms and kernel.

viii) Direct product. Some work on finite abelian groups. Tie-in with factorization of integers.

ix) Study of A_4 - no subgroups of order 6. Touch on Sylow Theory.

2. Rings and Fields.

i) From $(Z,+,\cdot)$, $(Q,+,\cdot)$, $(Z/(n),+_n,\cdot_n)$ → abstract ring. Role of zero, group theory.

ii) Direct sum Z x Z, Q x Q, coordinate-wise operations, zero divisors. Subrings, isomorphism, injection.

iii) Gaussian rationals. Field. Argand diagram.

iv) Gaussian integers. Divisor Theorem. "Binary" representation, using i + 1 as base. Geometric representation.

v) Euclidean algorithm. Primes and composites. Unique factorization. Sum of two squares.

vi) Ideals. "Sums" and intersection of ideals. Principal ideals. Maximal ideals and primes.

vii) Similar analysis of $Q[\sqrt{2}]$ and $Z[\sqrt{2}]$.

viii) 4-tuples and quaternions (rational).

ix) 8-tuples and Cayley algebras (see A.M.M. article by C. W. Curtis).

x) Ideals versus subrings. Difference (factor) rings. Homomorphism and kernel. Z/(n).

3. Polynomial Domains.

i) ∞-tuples, F x F x . . . -- all sequences over F -- all functions from W → F, F an entire ring. Define addition and multiplication to obtain polynomial ring, F[x]. Distinguish between polynomial ring and ring of polynomial functions.

ii) Consider $Q[x]$. Divisor Theorem. Specialization to Q and zeros. Remainder and Factor Theorems.

iii) Euclidean algorithm. Primes (irreducibles).

iv) $Q[x]$ and $Z[x]$. Rational zeros. Gauss' Lemma. Eisenstein's Theorem. Lagrange interpolation.

4. Euclidean Domains.

i) Abstract to definition.

ii) Conclusions about $F[x]$, where F is an arbitrary field.

iii) Principal ideal domains.

iv) Unique factorization domain.

v) Non-unique factorization rings: $Z[\sqrt{5}]$, $Z\sqrt{-5}]$

5. Finite Fields.

i) $F[x]$ where $F = Z/(p)$, p prime. Every function $F \to F$ is a polynomial function. Ring of polynomial functions finite, Fermat's Theorem.

ii) Splitting fields of $x^4 - x$ over GF(2), $x^9 - x$ over GF(3).

iii) Nature of finite field. Characteristic and prime fields.

iv) Simple applications in design of experiments, Latin squares, finite projective geometry. (H. B. Mann - "Finite Combinatorial Analysis and Design of Experiments"; Albert - "Finite Fields"; M. Hall - "Combinatorial Analysis").

- BOOK 12[1] -

LINEAR ALGEBRA AND TRIGONOMETRY

The main objective of this book should be a thorough investigation of the plane and space geometry using the language and tools of linear algebra. Thus emphasis will be on the development of two and three dimensional vector spaces over the field $\mathbb{R}$ of real numbers and the use of these vector spaces to study geometry. CSMP students will have studied fields other than the field of real numbers; therefore vector spaces can be defined over arbitrary fields with no restriction on the dimension. Examples and exercises might well be given about vector spaces over the field of complex numbers, the field of rational numbers and finite fields; nevertheless, the main investigation should be restricted to two and three dimensional vector spaces over the field of real numbers. If students obtain a thorough knowledge and working facility in these dimensions, the transition to higher dimensions is almost trivial. Similar remarks hold for the study of geometry.

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[1] This is the author's summary of Book 12 which is presently being written.

The outline is divided into the six "chapters" listed below.

1. Vector Spaces and Linear Transformations.
2. Affine Spaces and Affine Transformations.
3. Bilinear Functions (Inner Products).
4. Euclidean Geometry and Euclidean Transformations.
5. Trigonometry.
6. Further Topics from Linear Algebra and Geometry.

Two approaches will be used in the chapter outlines. When the material described is relatively standard, only topic headings will be listed. If the material described is not so standard, some definitions and a few typical propositions will be given in order to better indicate the approach and level of development.

CHPATER 1: VECTOR SPACES AND LINEAR TRANSFORMATIONS

1. Definition of Vector Space V.
2. Linear Subspaces.
3. Linear Combinations of Vectors; the Subspace Generated by a Set of Vectors.

4. Intersection and Sum of Subspaces.
5. Direct Sum of Subspaces.
6. Independence and Dependence of a Set of Vectors.
7. Bases and Dimension.
8. Linear Transformations.
9. Linear Isomorphism; Linear Automorphism.
10. Matrix Representations of a Linear Transformation.
11. Kernel and Image of a Linear Transformation.
12. Dimension Proposition Relating the
    a) Sum and Intersection of Two Linear Subspaces
    b) Kernel and Image of a Linear Transformation.

CHAPTER 2: AFFINE GEOMETRY AND AFFINE TRANSFORMATIONS

1. Affine Space.

Definition: Real affine space is a pair (V,X) where

a. V is a vector space over the field R of real numbers.

b. X is a nonempty set of objects called points.

c. There is a function $(A,x) \to Ax$ from $V \times X \to X$ satisfying

   i) $(A + B)x = A(Bx)$ for all $A,B \in V$ and all $x \in X$.

ii) For every ordered pair (x,y) of points in X, there is a unique vector $A \in V$ such that $Ax = y$.

2. The Vector Space of Translations.

Definition: For each vector $A \in V$, the function T_A: $X \to X$ defined by $T_A(x) = Ax$ is called a translation.

Proposition: The set $\mathfrak{T}$ of translations becomes a vector space over the field of real numbers by defining

$$T_A + T_B = T_{A+B} \quad \text{and} \quad rT_A = T_{rA}$$

Moreover, $\mathfrak{T}$ is isomorphic to V.

3. Affine Subspaces of (V,X).

Definition: Let $x \in X$ and U be a linear subspace of V. The set

$$S(U,x) = \{Ax \mid A \in U\}$$

is called the affine subspace determined by x and U. The dimension of $S(U,x)$ is defined to be the dimension of U.

Terminology:

1-dimensional affine subspaces are called lines.

2-dimensional affine subspaces are called planes.

$(n-1)$-dimensional affine subspaces are called hyperplanes (where $n = \dim V$).

Proposition: The affine subspaces $S(U,x)$ and $S(W,y)$ are equal if and only if $U = W$ and $S(U,x) \cap S(W,y) \neq \emptyset$.

Proposition: The pair $(U,S(U,x))$ is an affine space.

Proposition: If $z \in S(U,x) \cap S(W,y)$, then $S(U,x) \cap S(W,y) = S(U \cap W,z)$.

4. Coordinates for Affine Subspaces.

5. Analytic Geometry.

6. Parallelism.

Definition: The affine subspaces $S(U,x)$ and $S(W,y)$ are parallel if $U \subset W$ or $W \subset U$.

Notation: $S(U,x) \parallel S(W,y)$.

Proposition: If $S = S(U,x)$ and $S' = S(W,y)$ are affine subspaces of the same dimension, then

a) $S \parallel S'$ if and only if $U = W$.

b) $S \parallel S'$ if and only if there is a translation T such that $T(S) = S'$.

c) $S \parallel S' \Leftrightarrow S \cap S' = \emptyset$ or $S = S'$.

Proposition: (Fifth Parallel Axiom) Let $p \in X$ and $S = S(x,U)$ an affine subspace of X. Then there exists a unique affine subspace S' of X such that

a) $\dim S = \dim S'$.

b) $p \in S'$.

c) $S \parallel S'$.

7. Affine Subspace Spanned by Points.

8. Orientation; Betweenness; Simplexes.

9. Affine Transformations and Their Properties.

10. The Affine Group.

CHAPTER 3: BILINEAR FUNCTIONS (INNER PRODUCT SPACES)

1. Definition of Bilinear Function.
   (Here it is assumed that we only investigate the positive definite inner product of a vector space V over the field of real numbers.)

2. Definition of Orthogonal Vectors in V.

3. Definition of Orthogonal Subspaces.

4. The Orthogonal Complement U^* of a Linear Subspace U of V.

5. Orthogonal and Orthonormal Coordinate Systems of V.

6. The Gram-Schmidt Orthogonalization Process.

7. Isometries of V.

8. The Witt Theorem in V.
   Theorem: If U and U' are subspaces of V and

f: U → U' is an isometry, then f can be extended to an isometry $\overline{f}$: V → V.

9. The Orthogonal Group and the Rotation Group of V.

10. The Cartan-Dieudonné Theorem.
Theorem: Each isometry of V is the product of at most n reflections in hyperplanes of V, where dim V = n.

11. The Rotation Group when V is a Plane.

12. The Orthogonal Group when V is a Plane.

13. Rotations and Reflections when V is 3-Space.

14. The Orthogonal Group when V is 3-Space.

CHAPTER 4: EUCLIDEAN GEOMETRY AND EUCLIDEAN TRANSFORMATIONS

1. Euclidean Space.
Definition: Euclidean space is affine space (V,X) where V has the additional structure of a positive definite inner product on V.
Definition: If x,y ∈ X and A is the vector in V such that Ax = y, then the distance beween x and y is defined by the formula

$$d(x,y) = \sqrt{a^2}.$$

2. Euclidean Transformations (Isometries) of X.

3. The Euclidean Group.

Proposition: Select $c \in X$. The set of all Euclidean transformations of X which leave c fixed is a group isomorphic to the orthogonal group of X.

Proposition: Select $c \in X$. Every Euclidean transformation f of X can be expressed uniquely as a product $T\sigma$, where T is a translation of X and σ is a rigid motion of X which leaves c fixed.

4. The Cartan-Dieudonné Theorem in X.

Theorem: Every Euclidean transformation of X is the product of at most n + 1 reflections in hyperplanes of X.

5. Congruence.

Definition: A nonempty subset of X is called a figure.

Definition: Two figures F_1 and F_2 are called congruent if there is a Euclidean transformation σ of X such that $\sigma(F_1) = F_2$.

Proposition: Two traingles are congruent if and only if there is a correspondence of the vertices and sides such that corresponding sides are congruent (SSS).

Proposition: Two triangles are congruent if two sides and the included angle of one triangle are congruent to two sides and the included angle of the second triangle (SAS).

6. Classification of Euclidean Transformations of X as Translations, Rotations, Reflections and Glide Reflections.

7. Orthogonality in X.

8. Similarity in X.

CHAPTER 5: PLANE TRIGONOMETRY

1. Rotations and Angles.

2. Cosine of a Rotation.

3. Properties of the Cosine Function.

4. Orientation of the Plane, Counterclockwise and Clockwise.

5. Definition of the Sine of a Rotation.

6. Other Trigonometric Functions.

7. Trigonometric Formulas and Trigonometric Identities.

CHAPTER 6: FURTHER TOPICS FROM LINEAR ALGEBRA AND GEOMETRY

Several topics will be mentioned, but it is premature to make any decision which topics, if any, should be included in Book 12.

1. Solutions of Systems of Linear Equations.
2. Canonical Forms of Matrices.
3. Eigenvalues and Eigenvectors of a Linear Transformation.
4. Euclidean Space as a Topological Space; Continuous Transformations.
5. Convexity.

* * *

A total of about 15 books will comprise the EM series. The titles for Books 13 and 14 are:

Book 13: Introduction to Real Analysis I
Book 14: Introduction to Real Analysis II

Some additional books at more advanced levels will be prepared for students who finish Book 14 with a substantial amount of time left before high school graduation. These books will be used in a branching fashion with a student choosing that area which most interests him. Possibilities for topics to be covered by these books include:

- Combinatorics
- Probability Theory (infinite probability spaces)
- Algorithmic Theories and Computers
- Statistics
- Methodology of Applied Mathematics (with examples from the theory of magnitudes, rank orders, game theory, etc.)
- Axiomatic Set Theory (a brief discussion)
- Abstract Algebra (including some Galois theory)
- Structured Sets and Boolean Algebra

We do not intend rewriting all of mathematics. It is our aim to develop the students to the point where they can use the available good textbooks. There do not exist any elementary calculus books, however, that take advantage of the background these students will bring to the subject.

We anticipate that not all the students will have the time or interest for all the material that will be available, and that there will be some branching of programs in the last years. We know from experience, however, that some of our students will reach, in the high school years, to mathematics of a higher level than we intend to present. Such students will simply use existing textbooks, written at that level, until such time as they go to college.

At present, the kind of student who can take graduate courses in foundations of analysis or algebra as a freshman is extremely bright, with plenty of drive. Colleges will have to become used to, and deal properly in the future with, students who can start at this level not because they are in the gifted 1% tail of the distribution, but because they come from the upper 15-20% and are well educated in mathematics.

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APPENDIX VI

- Papers Presented at the March 1969 CSMP International Conference on the Teaching of Probability and Statistics
- Probability Conference Recommendations

APPENDIX VI

PAPERS PRESENTED AT THE MARCH 1969
CSMP INTERNATIONAL CONFERENCE
ON THE TEACHING OF PROBABILITY AND STATISTICS

E. F. BECKENBACH, University of California, Los Angeles, *Combinatorics for School Mathematics Curricula*.

C. B. BELL, University of Michigan, Ann Arbor, *Nonparametric Statistics at the Pre-College Level*.

H. DAVIES, University of Sheffield, England, *The Role of Practical Experimentation in the Teaching of Probability and Statistics*.

J. B. DOUGLAS, University of New South Wales, Kensington, Australia, *On the Teaching of Probability and Statistics at the Pre-College Level in Australia*.

A. ENGEL, University of Karlsruhe, West Germany, Teaching *Probability in Intermediate Grades*.

H. FREUDENTHAL, Mathematisch Instituut, Budapestlaan, Utrecht-Netherlands, *The Aims of Teaching Probability*.

J. GANI, University of Sheffield, England, *The Role of Mathematics in Conceptual Models*.

S. GOLDBERG, Oberlin College, Ohio, *Probability and Pre-School Analysis*.

P. L. HENNEQUIN, Université de Clermont, Clermont, France, *Teaching of Probability and Statistics in the French Lycee*.

S. HOLM, Chalmers Institute of Technology, Gothenberg, Sweden, *A Collection of Problems in Probability and Statistics*.

B. HUME, University of Western Australia, Nedlands, Australia, *The Introduction of Probability and Statistics on the Pre-College Level in Western Australia*.

W. KRUSKAL, University of Chicago, Illinois, Statistical Examples for Use in High School.

D. V. LINDLEY, University College, London, England, A Non-Frequentist View of Probability and Statistics.

J. NEYMAN, University of California, Berkeley, Elementary Teaching of Probability and Statistics with Indeterminism in Science as a Background.

L. RÅDE, Chalmers Institute of Technology, Gothenberg, Sweden, Relations, Functions and Expectations; and, Teaching of Probability and Statistics on the Pre-College Level in Sweden.

C. R. RAO, Indian Statistical Institute, Calcutta, India, A Multidisciplinary Approach for Teaching Statistics and Probability.

A. RENYI, Hungarian Academy of Sciences, Budapest, Hungary, Remarks on the Teaching of Probability.

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PROBABILITY CONFERENCE

RECOMMENDATIONS

1. We, the participants at the International Conference on the Teaching of Probability and Statistics, Carbondale, having received a brief but thorough introduction to the Comprehensive School Mathematics Program, wish to commend it as a highly interesting experiment in mathematical education. Any program whose aim is to bring school mathematics closer to current directions of mathematical usage and discovery performs a useful social function. The CSMP project, because of its breadth of approach, absence of dogmatism, flexibility and rich content of new didactic ideas (among them the individual package concept) deserves the most serious encouragement. The participants feel that it would be desirable to develop similar coordinated programs in other disciplines.

2. In designing their program, the CSMP staff would do well to bear in mind:
   (a) the need to stress the various applications of mathematics in the biological, physical and social sciences;

(b) the role of logic in mathematics, including the context of computer strategies;

(c) the role of the teacher in the implementation of the program;

(d) the desirability of handbooks for school mathematics curriculum designers, and teacher manuals as helpful supplements to the text and package material.

3. The participants strongly endorse CSMP's efforts to introduce probability and statistics as subjects for study at elementary and secondary school levels. They believe that these subjects should be taught starting from a wealth of realistic examples. Some emphasis should be placed on their use as tools, both for the development of mathematical structures and in the building of applied models.

4. In teaching probability, full advantage should be taken of practical experiments, and in particular of simulation methods. The knowledge acquired from such experiments should be directly reinforced by a theoretical framework; this should not be too rigid. In view of the different possible approaches to the subject, the formal concepts and theories presented

should be eclectic. Probability courses at the secondary level might include, in addition to the usual topics, the weak law of large numbers, some game theory and elementary stochastic processes.

5. Descriptive statistics of physical, biological and social data are subjects of great importance to every citizen. They can be taught at almost every level. Material of this kind could serve as an introduction to a school course which might include further topics in statistical theory and inference. Such a course should be taught in careful coordination with probability theory and should make use of realistic data wherever possible. Estimation, elementary tests of hypotheses, non-parametric statistics and decision theory might be suitable topics for inclusion in a course in statistics at the secondary level.

6. The participants greatly enjoyed the conference and they would appreciate continued contact with the CSMP Center at Carbondale. The dissemination of the program's results and material abroad, as well as in the U.S.A., would be most useful. Duplication of

effort would be avoided, and constructive comments and criticism from teachers and research workers all over the world would be encouraged.

The participants feel that there is a strong need for further CSMP Conferences in different fields of mathematics, to discuss content and curricula and to exchange teaching experiences and ideas. Part of the content of lectures and discussions in such conferences should be closely related to the work done at the Carbondale Center. The interest in the present conference was such that several of the participants expressed their willingness to act as advisors to CSMP at the discretion of its staff.

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APPENDIX VII

- Papers Presented at the March 1970 CSMP International Conference on the Teaching of Geometry
- Geometry Conference Recommendations

APPENDIX VII

PAPERS PRESENTED AT THE MARCH 1970 CSMP INTERNATIONAL CONFERENCE ON THE TEACHING OF GEOMETRY

F. BACHMAN, Christian-Albrechts-Universität, Kiel, West Germany, n-gons.

H. S. M. COXETER, University of Toronto, Ontario, Canada, Inversive Geometry.

Z. P. DIENES, Université de Sherbrooke, Québec, Canada, An Example of the Passage from the Concrete to the Manipulation of Formal Systems.

A. ENGEL, Universität Karlsruhe, West Germany, Geometrical Activities for the Upper Elementary School Grades.

T. J. FLETCHER, Department of Education and Science, Darlington, England, The Teaching of Geometry—Present Problems and Future Aims.

H. FREUDENTHAL, Mathematisch Instituut, Budapestlaan, Utrecht-Netherlands, Geometry Between the Devil and the Deep Sea.

P. J. HILTON, Cornell University, Ithaca, New York, Topology on the High School Level.

M. A. JEEVES, University of St. Andrews, St. Andrews, Fife, Scotland, Some Experiments on Structural Learning.

P. J. KELLY, University of California, Santa Barbara, Topology and Transformations in High School Geometry.

V. KLEE, University of Washington, Seattle, The Place of Unsolved Problems in the High School Geometry Curriculum.

H. LEVI, Herbert H. Lehman College, CUNY, Bronx, New York, Geometric Algebra for the High School Program.

K. MENGER, Illinois Institute of Technology, Chicago, Illinois, The Geometry Relevant for Modern Education.

G. PAPY, Université de Bruxelles, Belgium, First Notions of Topological Space.

G. PICKERT, Justus Liebig-Universität, Giessen, West Germany, The Introduction of a Metric by the Use of Conics.

A. REVUZ, Université de Paris, France, The Position of Geometry in Mathematical Education.

T. G. ROOM, University of Sydney, Australia, The Geometry and Algebra of Reflections.

S. SCHUSTER, Carleton College, Northfield, Minnesota On the Teaching of Geometry— A Potpourri.

D. S. SCOTT, Princeton University, New Jersey, The Algebra of Transformations.

H. G. STEINER, University of Karlsruhe, West Germany, The Role of Geometry in the CSMP Curriculum.

M. H. STONE, University of Massachusetts, Amherst, Learning and Teaching Axiomatic Geometry.

I. E. VANCE, Michigan State University, East Lansing, Some Comments on Geometry in the Elementary School.

H. E. VAUGHAN, University of Illinois, Urbana, The Development of Euclidean Geometry in Terms of Translations.

M. VILLA, Universitá di Bologna, Italy, The Rational Teaching of Geometry at the Secondary Level.

J. D. WILLIAMS, Université de Sherbrooke, Québec, Canada, Possibilities and Problems in the Assessment and Investigation of Mathematics Learning.

H. ZASSENHAUS, Ohio State University, Columbus, Genetic Development of the Congruence Axioms.

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GEOMETRY CONFERENCE

RECOMMENDATIONS

1. Geometry has evolved as an important means of understanding and organizing spatial phenomena. It is widely and multifariously applied in everyday life, whether in a highly mathematized form or not. Geometrical concepts, including their aesthetic aspects, have pervaded and elucidated other fields of mathematics. The richness of geometry, which lies in the wide variety of its concepts and problems, should be carefully exploited to give the student an anticipation of other branches of mathematics, which have either developed as powerful tools to study space (such as linear algebra, group theory) or as answers to questions that could be posed simply in geometric terms, but which could not be solved, or solved simply, within the framework of geometry (such as topology, measure theory).

2. The teaching of geometry should take into account the points stressed in the previous paragraph; in particular, geometry should play a part in the

teaching of ideas of conceptual model building. Moreover, the teaching of geometry should respect obvious facts of developmental psychology: it can start at an early age with concrete material in concrete situations; rather than being imposed according to a pre-conceived system, deductive structures should arise to the degree to which children are able to analyze situations. It depends on the maturity and on the gifts of students, as well as on the part attributed to geometry, to what extent geometry is deductively organized and axiomatized. As an organizing and classifying element in geometry, groups are essential and should be used extensively. As to the question of what geometrical subjects should be taught, one should free oneself from the traditional view of geometry, and carefully weigh subjects and aspects that have not been given sufficient attention in the past; for instance: incidence structures, convexity, transformations, metrics, topological ideas, order, orientation, etc.

3. Geometry should not be used primarily as a vehicle to teach logic; to do so would give a distorted view of

both subjects. Correct mathematics should be a goal of every activity, whether implications are drawn from informal exploration or from formal reasoning. The degree of rigor in the teaching of mathematics may vary according to circumstances, but that should never be an excuse to misinform or to mislead the student.

4. Integrated studies of mathematical curricula and teaching methods should be the concern of permanent centers, such as CSMP, employing the services of a full-time staff, regularly consulting with mathematicians of high standing, and organizing conferences that will bring together teachers and mathematicians from home and abroad.

   The effective teaching of geometry requires the use and the further development of suitable materials, such as apparatus for practical work, books and packages for individual and group use, and a wide range of visuals, including films. The conference commends CSMP for the admirable efforts which it has already made in this direction.

5. The mathematics taught in teacher training courses (and its geometric content, in particular) should not

be restricted to background knowledge: teachers should also be confronted with mathematical topics in the context in which they are to be taught. This means that courses in Advanced Mathematics from an Elementary Viewpoint should be designed with a view to the communication of elementary and advanced aspects of the subject matter and of its teaching. Insofar as the interrelation of geometry with other subjects has been neglected, a more balanced approach should be adopted.

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APPENDIX VIII

A: List of Recommended Courses for EM Teachers Institute

B: Elements of Mathematics Workshop

APPENDIX VIII.A

LIST OF RECOMMENDED COURSES FOR EM TEACHERS INSTITUTE

- Logic in Elementary Mathematics: This course develops formal logic through a first-order predicate calculus with equality and uses the logic to deal with formal aspects of elementary mathematics.

- Modern Abstract Algebra: Theory of groups, rings, fields, including the integers and polynomial rings.

- Topics in Mathematics: Seminar and readings in CSMP materials.

- Experimental Mathematics Program: Course will deal with mathematics content and the didactics of mathematics in experimental courses developed for high school students.

- Linear Algebra: Linear equations, finite dimensional vector spaces, linear dependence, transformations, matrices and determinants, eigenvectors, invariant subspaces, similarity.

- Set Theory: Essentially naive, as developed in CSMP materials.

- Probability: Foundations of probability restricted mainly to finite outcome sets. Independence relations, expectation, law of large numbers, Markov chains, and applications.

- Measurement and Evaluation Seminar: Seminar in problems of observation , measurement, evaluation of experimental programs.

- Foundations of Analysis I: Real number system, continuity, differentiability, sequences and series. Riemann and Stieltjies integrals, uniform convergence.

- Numerical Analysis and Computing: One week will be devoted to computer programming, Fortran IV, with an additional 25 hours of laboratory work on an IBM 360 computer.

- Topics in Mathematics: Seminar and readings in modern mathematics curriculum projects.

- Topology: Basic topological properties of plane point sets; axiomatic treatment of abstract topological spaces.

- Foundations of Analysis II: Continuation of course from Analysis I.

- Geometry.

- Statistics.

APPENDIX VIII.B

ELEMENTS OF MATHEMATICS WORKSHOP

(Copy of a Summer 1970 Program Brochure)

CSMP

The Comprehensive School Mathematics Program (CSMP), with headquarters in Carbondale, Illinois, is one of CEMREL's major programs. Its objective is to individualize mathematics instruction for all public and private school students, K through 12. Its activities are concentrated at three levels: selection and development of a sound mathematical content; development of activity packages for all students of ages 5 to 12; development of the Elements of Mathematics (EM) series for the upper 15-20 per cent of the secondary school student population.

THE EM SERIES

The Elements of Mathematics Program has been developed for a selected group of secondary school students who are usually characterized as "well motivated" and "of high verbal ability." The series appears presently in textbook form, but its presentation is designed to facilitate independent study. The complete series will comprise 15 books of rather sophisticated mathematics. (The list of titles is given in the overleaf; an asterisk marks the books that are available and that are presently being pilot tested.) An "Intuitive Background" volume, Book 0, which introduces the series, provides abundant mathematical experiences of an empirical and intuitive nature.

THE WORKSHOP

A teacher-training workshop will be held from June 22 till July 31, 1970, in the air-conditioned facilities of CSMP, Carbondale, Illinois. The purpose of the workshop is to train about 20 teachers who intend to use the EM materials in the fall of 1970.

The Elements of Mathematics series has been designed for students from the 7th through the 12th grade. Several options, however, are open for teachers who may not have the opportunity of covering the entire program.

Book 0, for instance, would make an excellent pre-algebra course for 7th and 8th graders. Books 0-4 could be used in grades 7 through 9 to treat (with a much more modern mathematical approach) the traditional Algebra I and II material, with the added assurance of an adequate preparation of the students for any junior or senior mathematics courses.

Teachers who plan to use the EM materials only at the junior high school level would most likely need to attend the workshop for one summer only. Additional workshops will be organized for teachers who plan to adopt the entire EM curriculum.

TUITION AND FEES

No tuition or fees will be charged by CSMP for the workshop. While no college credit is granted, CEMREL will issue a certificate to the participants indicating the successful completion of the two courses (each of which is approximately equivalent to three college credits) and of the seminar (equivalent to one college credit).

HOUSING ACCOMMODATIONS AND STIPENDS

CSMP, in cooperation with Southern Illinois University, will assist the participants in making housing arrangements. Single dormitory rooms cost $240 (plus tax) for 6 weeks (room/board); efficiency apartments are $120 - $135 monthly.

Stipends for participating teachers must come from the respective school systems or from outside sources, such as gifted student programs, Title III funds, etc.

COURSES OFFERED

1. AN OVERVIEW OF CEMREL/CSMP ELEMENTS OF MATHEMATICS PROGRAM.

   EM books 0, 1, 2, 3, and 4 will be used as basic texts with emphasis on gaining familiarity with the vocabulary and symbols used, on discussing pedagogical problems discovered in the use of the materials with the opportunity of observing a class of students working on Book 0.

2. PROBABILITY.

   Foundations of probability restricted mainly to finite outcome spaces, independence relations, expectation, law of large numbers, Markov chains, and applications. This course will be taught from an experimental approach and in the spirit of the EM curriculum.

3. MEASUREMENT AND EVALUATION SEMINAR.

   Seminar in problems of observation, measurement, and evaluation of experimental programs.

EM TEXTS

The entire EM series has been tentatively outlined through 15 of the books. The sequence is as follows:

*Book 0: - Intuitive Background
*Book 1: - Introductory Logic
*Book 2: - Logic and Sets
*Book 3: - Introduction to Field Theory
*Book 4: - Order in Fields
*Book 5: - An Introduction to Mathematization: A Theory of Voting Bodies
*Book 6: - Relations
*Book 7: - Functions
Book 8: - Number Systems
*Book 9: - Finite Probability Spaces
Book 10: - Elements of Geometry
Book 11: - Groups and Rings
Book 12: - Linear Algebra with Trigonometry
Book 13: - Introduction to Real Analysis I
Book 14: - Introduction to Real Analysis II